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Complex Hyperbolic Fuzzy Sets: Theory and Construction

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ABSTRACT

Uncertainty modeling has evolved from intuitionistic fuzzy sets to q-rung orthopair fuzzy sets, with successive extensions seeking to enlarge the admissible space for membership and non-membership evaluations. However, these conventional orthopair models remain bounded by restrictive linear or polynomial constraints, which may limit their ability to represent scenarios involving simultaneously high membership and non-membership degrees. Furthermore, standard orthopair models do not inherently represent periodicity, seasonality, or two-dimensional directional information. To address these limitations, this paper introduces the Complex Hyperbolic Fuzzy Set (Complex-HyFS), a novel framework that combines the algebraic flexibility of hyperbolic geometry with the vector-based representation of complex fuzzy sets. The Complex-HyFS is characterized by the hyperbolic constraint $r_O(x) \cdot r_P(x) \leq 1$, which geometrically extends the valid information space to the entire unit square in the amplitude plane, thereby permitting the independent assignment of optimistic and pessimistic degrees. Simultaneously, the integration of a complex phase term $e^{i\omega(x)}$ enables the representation of periodic uncertainty and constructive and destructive interference patterns. We rigorously establish the fundamental set-theoretic operations—union, intersection, and complement—and prove that the Complex-HyFS structure is closed under these operations. We further demonstrate that the proposed complement operator satisfies the extended De Morgan laws and preserves the hyperbolic constraint without degeneracy. Additionally, a systematic methodology for constructing Complex-HyFSs from classical fuzzy sets via complement generators is presented. Finally, the applicability of the framework is discussed in the contexts of bio-signal analysis and seasonal forecasting, illustrating the potential of Complex-HyFS as a generalized framework for modeling high-entropy and periodic uncertainty.

1. Introduction

The representation of uncertainty and vagueness in mathematical modeling has been a central pursuit in decision science, engineering, and artificial intelligence for over half a century. The seminal introduction of Fuzzy Sets (FS) by Zadeh [1] marked a paradigm shift from Aristotelian dual-valued logic to a multi-valued logic system, allowing elements to possess partial membership grades in the interval $[0, 1]$ [2]. While FS provided a robust framework for modeling linguistic imprecision [3, 4], it fundamentally lacked the capability to explicitly represent non-membership or rejection, assuming instead that non-membership is merely the complement of membership $(1 - \mu)$.

To address this limitation and capture the hesitation inherent in human decision-making, Atanassov [5] introduced the concept of Intuitionistic Fuzzy Sets (IFS). By defining separate membership (μ) and non-membership (ν) degrees subject to the linear constraint $\mu + \nu \leq 1$, IFS allowed for the modeling of indeterminacy ($\pi = 1 - \mu - \nu$). This generalization spurred extensive research in pattern recognition [6], medical diagnosis [7], and multi-criteria decision making (MCDM) [8, 9].

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However, the linear constraint of IFS proved restrictive in scenarios where decision-makers provide values that, while individually valid, sum to greater than one (e.g., $\mu = 0.7, \nu = 0.4$). To expand the feasible decision space, Yager [10, 11] proposed Pythagorean Fuzzy Sets (PFS), which relaxed the constraint to $\mu^2 + \nu^2 \leq 1$. This was further generalized by the introduction of q-rung Orthopair Fuzzy Sets (q-ROFS) by Yager [12] and Liu and Wang [13], where the constraint $\mu^q + \nu^q \leq 1$ ($q \geq 1$) allows for an increasingly broad space of admissible information, covering the entire unit square as $q \to \infty$. Subsequent extensions, such as Fermatean Fuzzy Sets ($q = 3$) [14] and various hybrid models [15, 16], have since been developed to fine-tune this granular uncertainty.

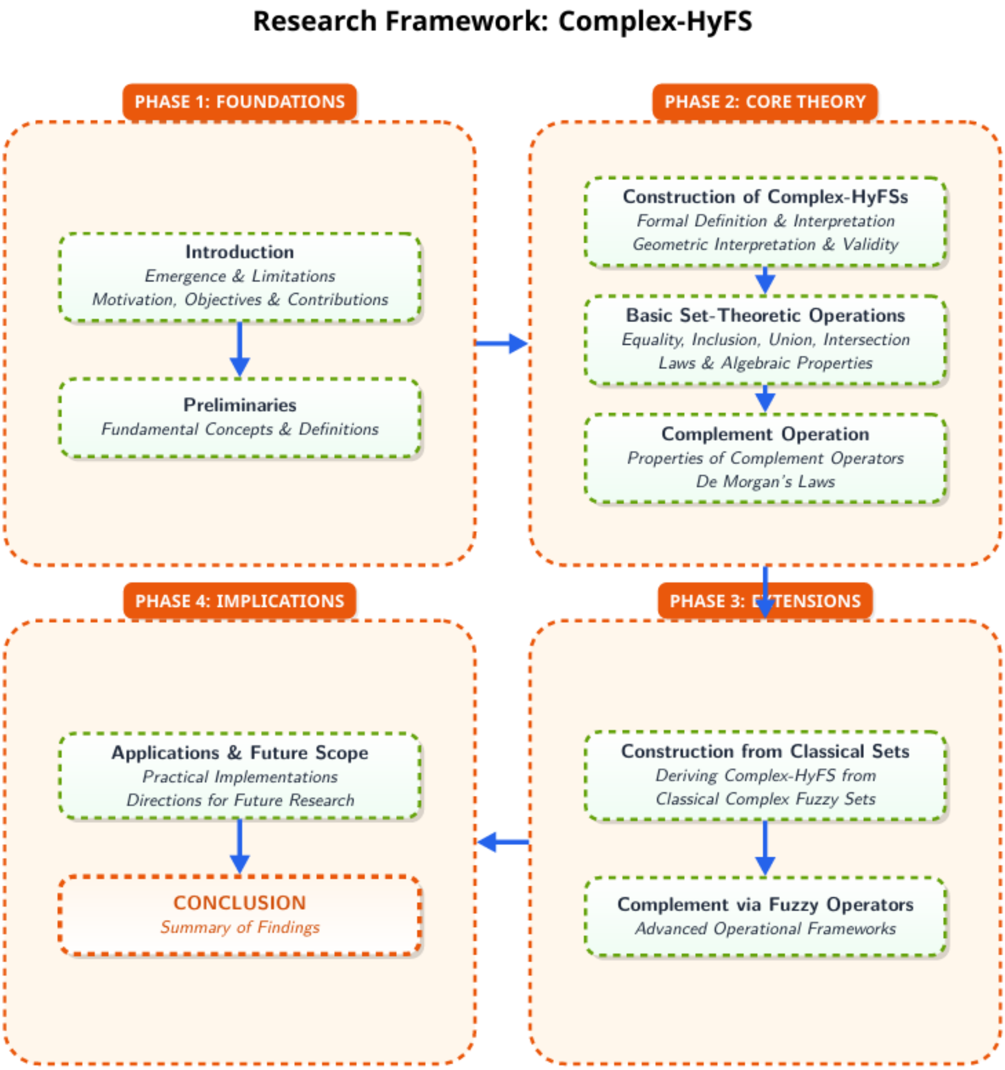

Fig. 1. The structural organization and logical progression of the proposed Complex-HyFS framework.

1.1 *The Emergence of Complex Fuzzy Theory*

Parallel to these developments in the real domain, a critical limitation was identified regarding the handling of periodic, seasonal, and two-dimensional uncertainty. Ramot et al. [17] argued that traditional fuzzy sets, being limited to the real line, cannot adequately model information that possesses a "phase" quality—such as the cyclical nature of solar activity [17], signal interference [18], or seasonal economic trends. To bridge this gap, they introduced Complex Fuzzy Sets (CFS).

In the CFS framework, the range of the membership function is extended from the real unit interval $[0, 1]$ to the unit circle in the complex plane. A complex membership grade is denoted as $\mu_S(x) = r_S(x) \cdot e^{i\omega_S(x)}$, where the amplitude $r_S(x)$ represents the magnitude of membership (uncertainty), and the phase term $\omega_S(x)$ represents periodicity or directional context [17]. This innovation allows for the modeling of constructive and destructive interference between membership grades, a feature akin to wave functions in

quantum mechanics [17, 19].
Following this, diverse extensions of complex fuzzy sets were proposed. Zhang et al. [20] explored the operational properties of CFS. In a landmark synthesis of orthopair and complex theories, Alkouri and Salleh [21] introduced Complex Intuitionistic Fuzzy Sets (CIFS). By assigning complex-valued functions to both membership and non-membership, CIFS enabled the simultaneous modeling of uncertainty and periodicity for both acceptance and rejection. This was subsequently extended to Complex Pythagorean Fuzzy Sets (CPFS) [22], Complex q-rung Orthopair Fuzzy Sets (Cq-ROFS) [23], and Complex Fermatean Fuzzy Sets [24], Complex Quintic fuzzy set (CQuFS) [25] establishing a comprehensive family of complex orthopair models.

1.2 Limitations of Existing Complex Orthopair Models

Despite the theoretical richness of the aforementioned models, they suffer from inherent algebraic and cognitive limitations that restrict their applicability in high-entropy decision environments:

1. **Restrictive Boundary Constraints:** The reliance on linear ($\mu + \nu \leq 1$) or polynomial ($\mu^q + \nu^q \leq 1$) constraints creates "dead zones" in the decision space. For instance, in CIFS, a decision-maker cannot express high simultaneous optimism and pessimism (e.g., $0.9, 0.9$) [21]. Even in Cq-ROFS, determining the appropriate q parameter can be computationally exhaustive and unintuitive for experts.

2. **Cognitive Dependency:** Current models inherently link the capacity for optimism to the capacity for pessimism. However, neuroscientific and behavioral studies suggest that human evaluation of "gain" (optimism) and "loss" (pessimism) occurs via independent neural pathways [26]. The standard orthopair constraints artificially force a trade-off between these dimensions.

3. **Phase Independence Issues:** In many existing complex extensions, the phase terms are often treated as secondary or are strictly bound to the amplitude's logic. There is a need for a framework where the periodicity of the complement is handled with the same rigor as the amplitude, satisfying De Morgan's laws without degeneracy [27, 28].

1.3 Motivation and The Hyperbolic Proposition

To address the boundary limitations of q-ROFS, recent work by Dutta et al. [26, 29] introduced the Hyperbolic Fuzzy Set (HyFS). The HyFS replaces the polynomial constraint with a hyperbolic condition: $O_\Lambda(x) \cdot P_\Lambda(x) \leq 1$. Geometrically, this condition ensures that the feasible region covers the entire unit square in the membership/non-membership plane. This allows for the representation of independent, ambivalent, and contradictory judgments without requiring parameter tuning (like q).
However, the standard HyFS is defined only in the real domain. It lacks the complex phase component necessary to model the temporal and periodic dynamics observed in bio-signals [30], and cyclic supply chain risks [31].
Therefore, the motivation of this research is to unify the *algebraic freedom* of Hyperbolic Fuzzy Sets with the *temporal richness* of Complex Fuzzy Sets. By generalizing the hyperbolic constraint to the complex plane, we can achieve a modeling framework that captures independent amplitudes (magnitude of belief/doubt) and independent phases (periodicity/timing) simultaneously.

1.4 Research Objectives and Contributions

The primary objective of this paper is to propose the Complex Hyperbolic Fuzzy Set (Complex-HyFS), a robust generalization that overcomes the limitations of both real-valued HyFS and conventional Complex Orthopair sets. The specific contributions are:

1. Formal Definition: We provide the mathematical definition of Complex-HyFS, establishing the hyperbolic constraint $r_O \cdot r_P \leq 1$ within the complex unit disk. This guarantees that the validity space encompasses the full unit square of amplitudes [26], distinguishing it from CIFS [21] and CPFS [22].

2. Set-Theoretic Operations: We construct the fundamental operations (Union, Intersection, Complement) for Complex-HyFS. Unlike previous approaches that rely on swapping membership/non-membership, we define independent complement generators that allow for mixed-type negation (e.g., combining Sugeno and Yager operators) while preserving the hyperbolic constraint.

3. Algebraic Properties: We rigorously prove that the Complex-HyFS framework satisfies the Extended De Morgan's Laws and the Involution property, ensuring algebraic consistency.

4. Construction Methodology: We outline a systematic method for constructing Complex-HyFS from classical fuzzy sets, bridging the gap between Zadeh's original theory [1] and this advanced hybrid model.

5. Applicability: We discuss potential applications in domains requiring periodic uncertainty handling, such as bio-signal analysis and seasonal forecasting, demonstrating the practical superiority of the model over restricted orthopair sets.

Organization of the Paper

This paper systematically develops the Complex Hyperbolic Fuzzy Set (Complex-HyFS) framework, proceeding from foundational theory to practical application. Section 1 delineates the research background, identifying the algebraic limitations of existing complex orthopair models and establishing the motivation for the hyperbolic approach. Section 2 reviews the requisite preliminaries, including Complex Intuitionistic, Pythagorean, Fermatean, Quartic, Quintic, and q-rung Orthopair fuzzy sets. Section 3 formulates the Complex-HyFS model, presenting its formal definition, cognitive interpretation, and the geometric validity of the hyperbolic constraint ($r_O \cdot r_P \leq 1$) within the complex unit disk. Section 4 details the core set-theoretic operations—equality, inclusion, union, and intersection—along with their algebraic properties. Section 5 constructs the specific complement operation for Complex-HyFS and provides a rigorous proof of its adherence to Extended De Morgan's Laws. Section 6 outlines a methodology for constructing Complex-HyFS from classical complex fuzzy sets via complement generators. Section 7 generalizes this construction using various fuzzy negation operators. Section 8 explores applications in bio-signal analysis and seasonal forecasting, while Section 9 concludes with a summary of contributions and future research avenues.

2. Preliminaries

In this section, we review the fundamental concepts of complex fuzzy set extensions that serve as the basis for the proposed framework. These concepts generalize classical fuzzy set theory by extending the range of membership from the real unit interval $[0, 1]$ to the unit disk in the complex plane, thereby enabling the modeling of periodicity and two-dimensional uncertainty.

Definition 1. (Complex Intuitionistic Fuzzy Set (CIFS) [21]) *A Complex Intuitionistic Fuzzy Set (CIFS) S defined on a universe of discourse Ω is characterized by a complex-valued membership function $\mu_S(x)$ and a complex-valued non-membership function $\nu_S(x)$, defined as:*

$$S = \{\langle x, \mu_S(x), \nu_S(x)\rangle : x \in \Omega\}$$

where

$$\mu_S(x) = r_\mu(x) \cdot e^{i\omega_\mu(x)},$$
$$\nu_S(x) = r_\nu(x) \cdot e^{i\omega_\nu(x)}.$$

The amplitudes $r_\mu(x), r_\nu(x) \in [0, 1]$ must satisfy the intuitionistic constraint:

$$0 \leq r_\mu(x) + r_\nu(x) \leq 1.$$

The phase terms $\omega_\mu(x)$ and $\omega_\nu(x)$ belong to $(0, 2\pi]$. The degree of indeterminacy (hesitancy) is given by $\pi_S(x) = 1 - r_\mu(x) - r_\nu(x)$.

Definition 2. (Complex Pythagorean Fuzzy Set (CPFS)[22]) *A Complex Pythagorean Fuzzy Set (CPFS) P on Ω is defined as an extension of CIFS where the restriction on the amplitudes is relaxed using the square sum. It is represented as:*

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle : x \in \Omega\}$$

subject to the condition:

$$0 \leq (r_\mu(x))^2 + (r_\nu(x))^2 \leq 1.$$

This allows for the representation of uncertainty where $r_\mu + r_\nu > 1$ but $r_\mu^2 + r_\nu^2 \leq 1$.

Definition 3. (Complex Fermatean Fuzzy Set (CFFS) [24]) *A Complex Fermatean Fuzzy Set (CFFS) F on Ω further relaxes the constraint by utilizing the cubic sum. It is defined as:*

$$F = \{\langle x, \mu_F(x), \nu_F(x)\rangle : x \in \Omega\}$$

subject to the condition:

$$0 \leq (r_\mu(x))^3 + (r_\nu(x))^3 \leq 1.$$

The CFFS can handle higher levels of uncertainty than CPFS (e.g., $r_\mu = 0.9, r_\nu = 0.6$) and its indeterminacy is given by $\pi_F(x) = \sqrt[3]{1 - (r_\mu(x)^3 + r_\nu(x)^3)}$.

Definition 4. (Complex Quartic Fuzzy Set (CQFS)) *A Complex Quartic Fuzzy Set (CQFS) K on Ω is defined for cases where the uncertainty requires a fourth-power constraint. It is represented as:*

$$K = \{\langle x, \mu_K(x), \nu_K(x)\rangle : x \in \Omega\}$$

with the constraint:

$$0 \leq (r_\mu(x))^4 + (r_\nu(x))^4 \leq 1.$$

This set accommodates wider information spaces than the Fermatean set.

Definition 5. (Complex q-rung Orthopair Fuzzy Set (Cq-ROFS) [23]) *The Complex q-rung Orthopair Fuzzy Set (Cq-ROFS) Q generalizes the aforementioned sets (where $q = 1, 2, 3, 4$) by introducing a variable parameter $q \geq 1$. It is defined as:*

$$Q = \{\langle x, \mu_Q(x), \nu_Q(x)\rangle : x \in \Omega\}$$

with the generalized constraint:

$$0 \leq (r_\mu(x))^q + (r_\nu(x))^q \leq 1.$$

The indeterminacy degree is given by $\pi_Q(x) = (1 - r_\mu(x)^q - r_\nu(x)^q)^{1/q}$. As q increases, the allowable space for membership and non-membership expands towards the unit square.

Definition 6. (Complex Quintic Fuzzy Set (CQuFS) [25]) *A Complex Quintic Fuzzy Set (CQuFS) is the specific higher-order extension where $q = 5$. It is particularly useful for highly uncertain decision-making environments where the orthopair satisfies:*

$$0 \leq (r_\mu(x))^5 + (r_\nu(x))^5 \leq 1.$$

The CQuFS provides a wider valid information space than the Pythagorean, Fermatean, and Quartic fuzzy sets, capturing finer granularities of uncertainty.

Figure 2 presents a geometric comparison of the permissible amplitude regions across six distinct classes of complex fuzzy sets. Subplots (a) through (e) illustrate the progressively expanding boundaries of the Complex Intuitionistic ($q = 1$), Complex Pythagorean ($q = 2$), Complex Fermatean ($q = 3$), Complex Quartic ($q = 4$), and Complex Quintic ($q = 5$) fuzzy sets and complex-HyFS, respectively.

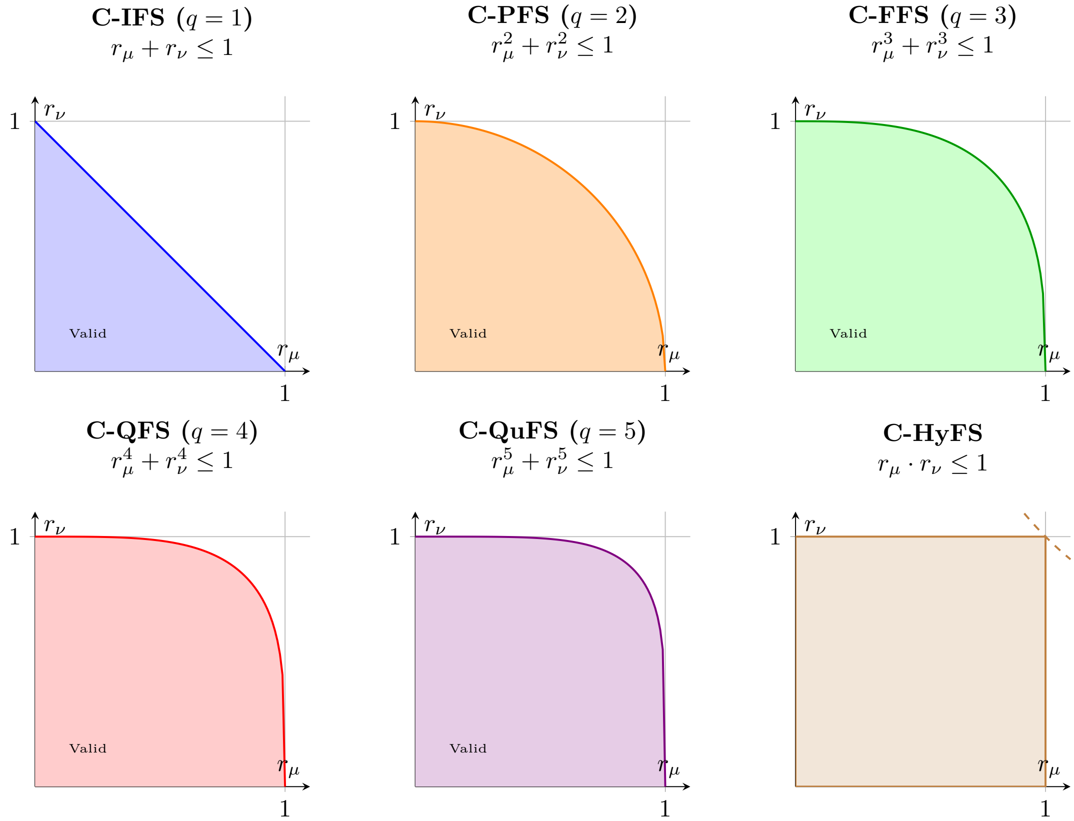

Fig. 2. Geometric comparison of the valid amplitude regions for six classes of complex fuzzy sets.

3. Construction of Complex-HyFSs

In this section, we extend the concept of Hyperbolic Fuzzy Sets (HyFS), originally proposed by Dutta [26], to the complex domain. This extension, termed Complex Hyperbolic Fuzzy Sets (Complex-HyFS), incorporates an additional phase term to model periodicity and two-dimensional uncertainty, while preserving the semantic flexibility of optimistic and pessimistic degrees.

The Complex Hyperbolic Fuzzy Set (Complex-HyFS) emerges from a geometric reinterpretation of the general conic section—specifically the hyperbola—extended into the complex plane. This formulation provides a natural mathematical basis for independently expressing the optimistic (membership) and pessimistic (non-membership) degrees associated with an element $x \in \Omega$, considering both their magnitude and their periodic phase.

The derivation of the Complex-HyFS amplitude constraint follows from the geometric equation of a hyperbola and its transformation under coordinate rotation.

Let us consider the standard hyperbola given by:

$$\frac{x^2}{2} - \frac{y^2}{2} = 1.$$

Let a point $P(x, y)$ lie on this hyperbola, where x and y denote the projections of P on the respective axes. Expressing these coordinates in polar form, we have $x = r\cos\Theta$ and $y = r\sin\Theta$, where r is the radius vector and Θ is the inclination angle. If the coordinate system is rotated by $45°$, the new coordinates of the point P with respect to the rotated axes (x', y') are obtained through the rotation transformation:

$$x' = r\cos(45° - \Theta) = \frac{r}{\sqrt{2}}(\cos\Theta + \sin\Theta),$$

$$y' = r\sin(45° - \Theta) = \frac{r}{\sqrt{2}}(\cos\Theta - \sin\Theta).$$

Substituting these expressions into the original equation of the hyperbola yields:

$$x'y' = 1.$$

This can be generalized as $xy = 1$, which represents a rectangular hyperbola symmetric about both coordinate axes.
In the Complex-HyFS formulation, the feasible region of interest for the amplitudes is bounded by the curve $xy = 1$, or equivalently $xy \leq 1$. Each coordinate pair (x, y) within this region signifies the admissible combination of the *magnitudes* of membership and non-membership. Interpreting this geometrically, the boundary $xy = 1$ represents the maximum attainable product between the two attributes.
Replacing (x, y) with the amplitudes of the complex optimistic and pessimistic functions, $r_O(x)$ and $r_P(x)$, the fundamental condition of a Complex Hyperbolic Fuzzy Set is expressed as:

$$r_O(x) \cdot r_P(x) \leq 1, \qquad \forall x \in \Omega.$$

Since the amplitudes r_O and r_P are normalized within $[0, 1]$, their product is always ≤ 1. Thus, the Complex-HyFS covers the entire unit square in the amplitude plane.

3.1 *Formal Definition and Cognitive Interpretation*

The nomenclature *"Complex Hyperbolic Fuzzy Set"* arises from this derivation, as the amplitude condition corresponds to the region enclosed by the rotated hyperbola. Compared with Complex q-ROFS and its extensions, Complex-HyFS permits a broader mapping of the membership domains. This independence allows decision-makers to assign optimistic and pessimistic values simultaneously without violating algebraic constraints.

Definition 7. *Let Ω be a universe of discourse. A Complex-HyFS Λ is defined as:*

$$\Lambda = \{\langle x, O_\Lambda(x), P_\Lambda(x)\rangle : x \in \Omega\},$$

where $O_\Lambda(x)$ and $P_\Lambda(x)$ denote the complex optimistic and complex pessimistic degrees, defined by:

$$O_\Lambda(x) = r_O(x) \cdot e^{i\omega_O(x)}, \quad P_\Lambda(x) = r_P(x) \cdot e^{i\omega_P(x)}.$$

The amplitudes satisfy the trivial hyperbolic condition:

$$0 \leq r_O(x) \cdot r_P(x) \leq 1,$$

and the phase terms $\omega_O(x), \omega_P(x)$ lie within $[0, 2\pi]$. In this formulation, the conventional notions of membership are reinterpreted as optimistic *and* pessimistic *degrees, enriched by the phase term which captures periodicity or temporal context.*

Cognitive Justification: The adoption of optimistic and pessimistic degrees represents a paradigm shift grounded in empirical decision science. This terminology accurately reflects cognitive processes where positive opportunities (optimistic assessment) and negative constraints (pessimistic assessment) are evaluated through independent neural pathways. Neuroeconomic evidence demonstrates distinct processing systems for potential gains (ventral striatum activation) versus potential losses (amygdala activation).
The Complex-HyFS framework aligns with these principles:

- **Amplitudes (r_O, r_P):** Model the *intensity* of activation in these distinct neural pathways. The hyperbolic constraint allows for high simultaneous activation (ambivalence), which is psychologically valid.
- **Phases (ω_O, ω_P):** Model the *temporal context* or *periodicity* of these assessments (e.g., optimism regarding a short-term gain vs. a long-term gain), adding a necessary dimension for dynamic decision modeling.

Unlike the q-ROFS or (n, m)-ROFS families which introduce explicit hesitation degrees to manage constraints, the multiplicative constraint of Complex-HyFS inherently accommodates total uncertainty within the unit square. Consequently, Complex-HyFS is capable of capturing paradoxical, ambivalent, or contradictory assessments without loss of information, enhancing its applicability in complex domains such as medical diagnosis and risk analysis.

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3.2 Geometric Interpretation and Validity

The introduction of the condition $0 \leq r_O(x) \cdot r_P(x) \leq 1$ distinguishes the Complex-HyFS from existing generalizations such as Complex Intuitionistic Fuzzy Sets (CIFS) and Complex Pythagorean Fuzzy Sets (CPFS).

Remark 1. *In the formulation of CIFS, the constraint is linear ($r_\mu + r_\nu \leq 1$), restricting the valid space to a triangle. In CPFS, the constraint is quadratic ($r_\mu^2 + r_\nu^2 \leq 1$), restricting the space to a quarter circle. In contrast, the Complex-HyFS formulation utilizes the hyperbolic condition derived from the curve $xy = 1$. Since the individual amplitudes $r_O(x)$ and $r_P(x)$ are normalized in the interval $[0, 1]$, their product is mathematically guaranteed to be less than or equal to 1. Consequently, the valid region for the amplitudes of a Complex-HyFS covers the entire unit square in the Cartesian plane.*

This expansion allows the Complex-HyFS to model scenarios where the optimistic and pessimistic degrees are independently high (e.g., $r_O = 0.9, r_P = 0.9$), a situation which is considered invalid in both CIFS and CPFS frameworks. The phase terms $\omega_O(x)$ and $\omega_P(x)$ remain independent, allowing the model to capture distinct temporal or periodic aspects of optimism and pessimism.

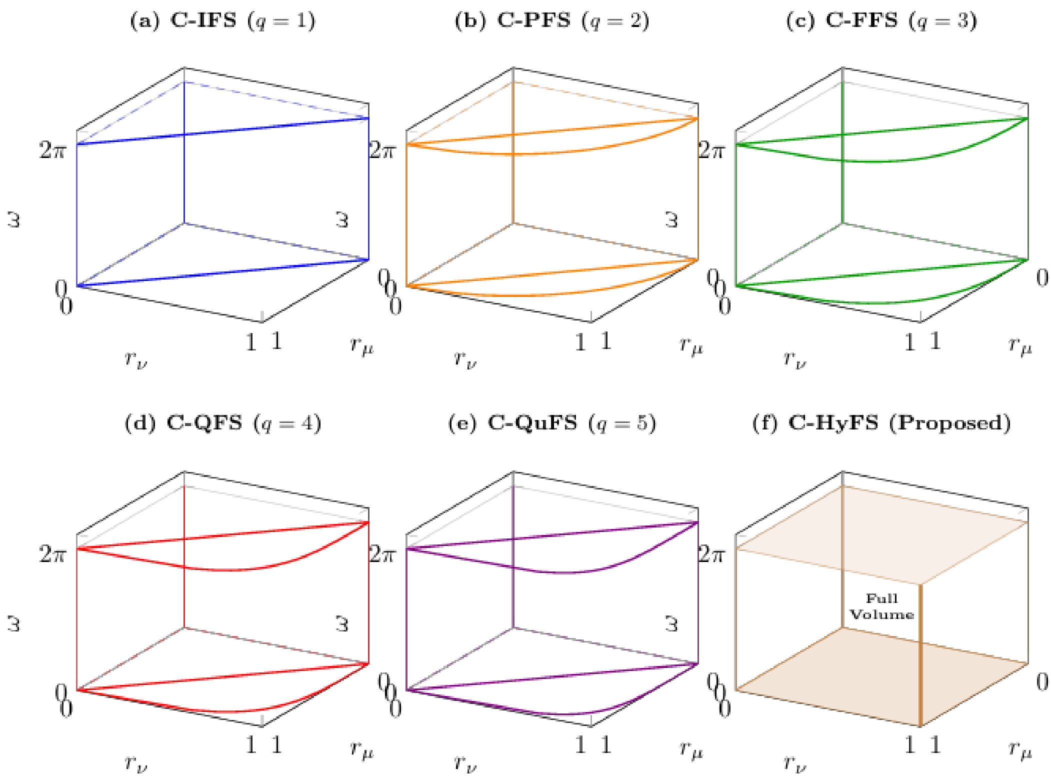

Fig. 3. 3D volumetric comparison of valid information spaces.

As demonstrated in Figure 3, the valid information space for complex fuzzy sets can be visualized as a 3D volume where the z-axis represents the phase periodicity $\omega \in [0, 2\pi]$. Subplots (a) through (e) illustrate the geometric limitations of existing orthopair fuzzy sets, where the permissible combinations of optimistic (r_μ) and pessimistic (r_ν) degrees are confined to restricted shapes such as triangular prisms or quarter-cylinders. In contrast, Subplot (f) reveals that the proposed Complex Hyperbolic Fuzzy Set (C-HyFS) framework utilizes the entire square prism volume defined by $[0, 1] \times [0, 1] \times [0, 2\pi]$. This volumetric expansion confirms that C-HyFS provides the maximum representational capacity, allowing for independent assignment of high membership and non-membership values without being constrained by mutual exclusion principles found in earlier theories.

Theorem 1. *Every Complex q-rung Orthopair Fuzzy Set (Cq-ROFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not generally true. Let $\mathcal{C}_q$ denote the class of all Cq-ROFSs (where $q \geq 1$) and $\mathcal{C}_\mathcal{H}$ denote the class of all Complex-HyFSs. Then, $\mathcal{C}_q \subset \mathcal{C}_\mathcal{H}$.*

Proof. Let Q be a Cq-ROFS with amplitudes $r_\mu, r_\nu \in [0,1]$ satisfying the condition $r_\mu^q + r_\nu^q \leq 1$. We apply the Arithmetic Mean-Geometric Mean (AM-GM) inequality to the terms r_μ^q and r_ν^q:

$$\sqrt{r_\mu^q \cdot r_\nu^q} \leq \frac{r_\mu^q + r_\nu^q}{2}.$$

Substituting the Cq-ROFS constraint $r_\mu^q + r_\nu^q \leq 1$ into the inequality:

$$\sqrt{(r_\mu r_\nu)^q} \leq \frac{1}{2}.$$

Squaring both sides yields:

$$(r_\mu r_\nu)^q \leq \frac{1}{4}.$$

Taking the q-th root (where $q \geq 1$):

$$r_\mu r_\nu \leq \left(\frac{1}{4}\right)^{1/q}.$$

Since $q \geq 1$, the term $(0.25)^{1/q}$ is always strictly less than 1. Therefore, the hyperbolic condition $r_\mu \cdot r_\nu \leq 1$ holds for all $q \geq 1$. Thus, $\mathcal{C}_q \subseteq \mathcal{C}_\mathcal{H}$.
To prove the strict inclusion ($\mathcal{C}_\mathcal{H} \not\subseteq \mathcal{C}_q$), consider the counterexample where $r_\mu = 1$ and $r_\nu = 1$.

- For Complex-HyFS: $1 \cdot 1 = 1 \leq 1$ (Valid).
- For Cq-ROFS: $1^q + 1^q = 1 + 1 = 2 > 1$ (Invalid for any finite q).

This demonstrates that there are elements in the Complex-HyFS space that do not belong to the Cq-ROFS space. Hence, $\mathcal{C}_q \subset \mathcal{C}_\mathcal{H}$. ∎

Proposition 2. *Every Complex Intuitionistic Fuzzy Set (CIFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not true.*

Proof. Let A be a CIFS with amplitudes $r_\mu, r_\nu \in [0,1]$ satisfying $r_\mu + r_\nu \leq 1$. According to the AM-GM inequality, $\sqrt{r_\mu r_\nu} \leq \frac{r_\mu + r_\nu}{2}$. Substituting the CIFS constraint, we get $\sqrt{r_\mu r_\nu} \leq 0.5$, which implies $r_\mu r_\nu \leq 0.25$. Since $0.25 < 1$, the hyperbolic condition $r_\mu \cdot r_\nu \leq 1$ holds. However, the converse fails as demonstrated by the point $(0.9, 0.9)$, which is valid in Complex-HyFS ($0.81 \leq 1$) but invalid in CIFS ($1.8 > 1$). ∎

Proposition 3. *Every Complex Pythagorean Fuzzy Set (CPFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not true.*

Proof. Let P be a CPFS satisfying $r_\mu^2 + r_\nu^2 \leq 1$. Using the inequality $2ab \leq a^2 + b^2$, we have $2r_\mu r_\nu \leq r_\mu^2 + r_\nu^2 \leq 1$, which implies $r_\mu r_\nu \leq 0.5$. Since $0.5 < 1$, the hyperbolic condition is satisfied. The converse is disproven by the counterexample $(0.8, 0.8)$, which is valid in Complex-HyFS ($0.64 \leq 1$) but invalid in CPFS ($1.28 > 1$). ∎

Proposition 4. *Every Complex Fermatean Fuzzy Set (CFFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not generally true.*

Proposition 5. *Every Complex Quartic Fuzzy Set (CQFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not generally true.*

Proposition 6. *Every Complex Quintic Fuzzy Set (CQuFS) is a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not generally true.*

Proposition 7. *Every Hyperbolic Fuzzy Set (HyFS) can be considered as a specific instance of a Complex Hyperbolic Fuzzy Set (Complex-HyFS), but the converse is not true. Formally, if $\mathcal{H}$ denotes the space of all HyFSs and $\mathcal{C}_\mathcal{H}$ denotes the space of all Complex-HyFSs, then $\mathcal{H} \subset \mathcal{C}_\mathcal{H}$.*

Proof. Part 1: HyFS $\implies$ Complex-HyFS
Let Λ be a Hyperbolic Fuzzy Set defined on a universe Ω. For any element $x \in \Omega$, the optimistic and pessimistic degrees are real-valued functions $O_\Lambda(x) \in [0, 1]$ and $P_\Lambda(x) \in [0, 1]$, satisfying the condition:

$$0 \leq O_\Lambda(x) \cdot P_\Lambda(x) \leq 1.$$

Any real number $r \in [0, 1]$ can be expressed in the complex plane as $r \cdot e^{i \cdot 0}$, where the amplitude is r and the phase is 0. We can construct a Complex-HyFS Λ' by assigning zero phase to both degrees:

$$\Lambda' = \{\langle x, O_\Lambda(x) \cdot e^{i0}, P_\Lambda(x) \cdot e^{i0}\rangle : x \in \Omega\}.$$

Here, the amplitudes are $r_O(x) = O_\Lambda(x)$ and $r_P(x) = P_\Lambda(x)$. Since the original HyFS condition holds ($O_\Lambda(x) \cdot P_\Lambda(x) \leq 1$), the amplitude condition for Complex-HyFS ($r_O \cdot r_P \leq 1$) is automatically satisfied. Thus, every HyFS is a Complex-HyFS with a phase vector of zero.

Part 2: Complex-HyFS $\not\Rightarrow$ HyFS (Converse)
Consider a Complex-HyFS Ψ defined by:

$$\Psi = \{\langle x, 0.5 \cdot e^{i\frac{\pi}{2}}, 0.5 \cdot e^{i\pi}\rangle\}.$$

The amplitudes 0.5 and 0.5 satisfy the hyperbolic constraint ($0.5 \times 0.5 = 0.25 \leq 1$). However, the membership values are:

$$O_\Psi(x) = 0.5(\cos(\pi/2) + i\sin(\pi/2)) = 0.5i \quad (\text{Purely Imaginary}),$$

$$P_\Psi(x) = 0.5(\cos(\pi) + i\sin(\pi)) = -0.5 \quad (\text{Negative Real}).$$

A standard HyFS requires degrees to be non-negative real numbers in $[0, 1]$. Since $O_\Psi(x) \notin \mathbb{R}$ and $P_\Psi(x) \notin [0, 1]$, Ψ cannot be represented as a standard HyFS. The presence of the phase term $\omega(x) \neq 0$ introduces periodicity and dimensionality that the real-valued HyFS framework cannot accommodate.

Thus, $\mathcal{H}$ is a proper subset of $\mathcal{C}_\mathcal{H}$. ■

3.3 Generalized Modal Operators in Complex-HyFS

Modal operators play a crucial role in transforming the granularity of information in fuzzy decision-making.
Unlike conventional orthopair sets where modal operators often map values to the boundary surface (e.g., $\mu + \nu = 1$), the Complex-HyFS valid space is the entire unit square $[0, 1] \times [0, 1]$. Therefore, we define the modal operators as **scaling transformations** that adjust the intensity of belief and doubt without violating the hyperbolic constraint.

Definition 8. (Generalized Parametric Modal Operator) *Let $\Lambda = \{\langle x, r_O(x)e^{i\omega_O(x)}, r_P(x)e^{i\omega_P(x)}\rangle : x \in \Omega\}$ be a Complex-HyFS. For any real numbers $\alpha, \beta \in [0, 1]$, the generalized modal operator $\mathbb{M}_{\alpha,\beta}$ is defined as:*

$$\mathbb{M}_{\alpha,\beta}(\Lambda) = \left\{\left\langle x, \alpha \cdot r_O(x)e^{i\omega_O(x)}, \beta \cdot r_P(x)e^{i\omega_P(x)}\right\rangle : x \in \Omega\right\}$$

This operator independently scales the optimistic and pessimistic amplitudes while retaining the original phase information.

3.4 Special Cases: Necessity and Possibility

From the generalized operator $\mathbb{M}_{\alpha,\beta}$, we derive the fundamental modal operators analogous to those in classical modal logic:

1. **Necessity Operator ($\Box$):** By setting $\alpha = 1$ and $\beta = 0$, we obtain the Necessity operator, which retains full optimism but eliminates pessimism:

$$\Box\Lambda = \mathbb{M}_{1,0}(\Lambda) = \{\langle x, r_O(x)e^{i\omega_O(x)}, 0\rangle\}$$

This represents a state of "certainty" regarding the positive evidence.

2. **Possibility Operator ($\Diamond$):** By setting $\alpha = 0$ and $\beta = 1$, we obtain the Possibility operator (in the context of rejecting the optimistic evidence):

$$\Diamond\Lambda = \mathbb{M}_{0,1}(\Lambda) = \{\langle x, 0, r_P(x)e^{i\omega_P(x)}\rangle\}$$

3. **Dilation Operator (Dil):** By setting $\alpha = \sqrt{r_O}$ and $\beta = \sqrt{r_P}$ (conceptually similar to $q = 0.5$), we can expand the uncertainty space. However, for linear scaling, we typically use fixed constants like $\alpha = \beta = 0.5$ to represent a "softening" of the stance.

3.5 Properties of Modal Operators

We establish that the Complex-HyFS framework is closed under these modal operations and satisfies key algebraic properties.

Theorem 8. (Closure Property) *For any Complex-HyFS Λ and parameters $\alpha, \beta \in [0, 1]$, the transformed set $\mathbb{M}_{\alpha,\beta}(\Lambda)$ is also a valid Complex-HyFS.*

Proof. Let the transformed amplitudes be $r'_O(x) = \alpha r_O(x)$ and $r'_P(x) = \beta r_P(x)$. The validity condition for Complex-HyFS is $r'_O(x) \cdot r'_P(x) \leq 1$. Substituting the definitions:

$$(\alpha r_O(x)) \cdot (\beta r_P(x)) = \alpha\beta(r_O(x) \cdot r_P(x)).$$

Since Λ is a valid Complex-HyFS, we know $r_O(x) \cdot r_P(x) \leq 1$. Given that $\alpha, \beta \in [0, 1]$, their product $\alpha\beta \leq 1$. Therefore:

$$\alpha\beta(r_O(x) \cdot r_P(x)) \leq 1 \cdot 1 = 1.$$

Thus, the hyperbolic constraint is preserved, and the closure property holds. ■

Theorem 9. (Monotonicity) *For a Complex-HyFS Λ, if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $0 \leq \beta_1 \leq \beta_2 \leq 1$, then:*

$$\mathbb{M}_{\alpha_1,\beta_1}(\Lambda) \subseteq \mathbb{M}_{\alpha_2,\beta_2}(\Lambda)$$

based on the amplitude inclusion order.

Proof. The inclusion $\Lambda_A \subseteq \Lambda_B$ is defined by $r_{O_A} \leq r_{O_B}$ and $r_{P_A} \leq r_{P_B}$ (assuming a non-reciprocal definition of inclusion for amplitudes). Since $\alpha_1 \leq \alpha_2$, it follows that $\alpha_1 r_O(x) \leq \alpha_2 r_O(x)$. Similarly, $\beta_1 r_P(x) \leq \beta_2 r_P(x)$. Thus, the monotonicity holds. ■

Theorem 10. (Idempotency of Identity) *If $\alpha = 1$ and $\beta = 1$, the operator acts as an identity function:*

$$\mathbb{M}_{1,1}(\Lambda) = \Lambda$$

Proof. Substituting $\alpha = 1, \beta = 1$ into Definition 5.1 yields $\langle 1 \cdot r_O e^{i\omega_O}, 1 \cdot r_P e^{i\omega_P}\rangle$, which is exactly Λ. ■

4. Basic Set-Theoretic Operations in Complex-HyFS

The fundamental set-theoretic operations—union, intersection, complement, inclusion, and equality—form the algebraic backbone of the *Complex Hyperbolic Fuzzy Set* (Complex-HyFS) framework. Each operation defined below is constructed to preserve the fundamental hyperbolic constraint:

$$0 \leq r_{O_\Lambda}(x) \cdot r_{P_\Lambda}(x) \leq 1,$$

ensuring that the interaction between the optimistic and pessimistic amplitudes remains within the valid unit square, while simultaneously managing the periodic phase terms.

Let Λ_1 and Λ_2 be two Complex-HyFSs defined on the same universe of discourse Ω, represented as:

$$\Lambda_k = \{\langle x, r_{O_k}(x)e^{i\omega_{O_k}(x)}, r_{P_k}(x)e^{i\omega_{P_k}(x)}\rangle : x \in \Omega\}, \quad k = 1, 2.$$

The operations between Λ_1 and Λ_2 are defined as follows:

4.1 Equality and Inclusion

Definition 9. *Two Complex-HyFSs Λ_1 and Λ_2 are said to be* equal, *denoted as $\Lambda_1 = \Lambda_2$, if and only if both their amplitude and phase terms are identical:*

$$r_{O_1}(x) = r_{O_2}(x), \quad \omega_{O_1}(x) = \omega_{O_2}(x),$$

$$r_{P_1}(x) = r_{P_2}(x), \quad \omega_{P_1}(x) = \omega_{P_2}(x), \quad \forall x \in \Omega.$$

Definition 10. *A Complex-HyFS Λ_1 is said to be* contained *in another Complex-HyFS Λ_2, denoted as $\Lambda_1 \subseteq \Lambda_2$, if the amplitudes satisfy the condition of dominance:*

$$r_{O_1}(x) \leq r_{O_2}(x) \quad \textit{and} \quad r_{P_1}(x) \geq r_{P_2}(x), \quad \forall x \in \Omega.$$

This definition implies that Λ_2 represents a state of higher optimistic magnitude and lower pessimistic magnitude compared to Λ_1. The inclusion relation preserves the hyperbolic constraint, ensuring that if $\Lambda_1 \subseteq \Lambda_2$, the product of the resulting amplitudes in Λ_2 remains ≤ 1.

Note: In many complex fuzzy frameworks, inclusion is defined primarily on the amplitude terms, as the phase terms (ω) represent periodicity or context rather than magnitude of truth.

4.2 Union Operation

Definition 11. *The* union *of two Complex-HyFSs Λ_1 and Λ_2 is a Complex-HyFS, denoted by $\Lambda_1 \cup \Lambda_2$, defined as:*

$$\Lambda_1 \cup \Lambda_2 = \left\{ \left\langle x, \max(r_{O_1}, r_{O_2}) e^{i \cdot \max(\omega_{O_1}, \omega_{O_2})}, \min(r_{P_1}, r_{P_2}) e^{i \cdot \min(\omega_{P_1}, \omega_{P_2})} \right\rangle : x \in \Omega \right\}.$$

The union operation satisfies the following properties:

- **Commutativity:** $\Lambda_1 \cup \Lambda_2 = \Lambda_2 \cup \Lambda_1$.
- **Associativity:** $(\Lambda_1 \cup \Lambda_2) \cup \Lambda_3 = \Lambda_1 \cup (\Lambda_2 \cup \Lambda_3)$.
- **Idempotence:** $\Lambda_1 \cup \Lambda_1 = \Lambda_1$.
- **Identity Element:** $\Lambda \cup \emptyset = \Lambda$, where $\emptyset = \{\langle x, 0e^{i0}, 1e^{i0} \rangle\}$ represents the null Complex-HyFS.

The use of the $\max$ operator for the optimistic terms and $\min$ for the pessimistic terms guarantees that the resulting amplitudes satisfy $r_{O_\cup} \cdot r_{P_\cup} \leq 1$. Furthermore, the phase aggregation ensures that the union reflects the most advanced periodic state of optimism and the earliest state of pessimism.

4.3 Intersection Operation

Definition 12. *The* intersection *of two Complex-HyFSs Λ_1 and Λ_2 is defined as:*

$$\Lambda_1 \cap \Lambda_2 = \left\{ \left\langle x, \min(r_{O_1}, r_{O_2}) e^{i \cdot \min(\omega_{O_1}, \omega_{O_2})}, \max(r_{P_1}, r_{P_2}) e^{i \cdot \max(\omega_{P_1}, \omega_{P_2})} \right\rangle : x \in \Omega \right\}.$$

The intersection operation satisfies the following properties:

- **Commutativity:** $\Lambda_1 \cap \Lambda_2 = \Lambda_2 \cap \Lambda_1$.
- **Associativity:** $(\Lambda_1 \cap \Lambda_2) \cap \Lambda_3 = \Lambda_1 \cap (\Lambda_2 \cap \Lambda_3)$.
- **Idempotence:** $\Lambda_1 \cap \Lambda_1 = \Lambda_1$.
- **Identity Element:** $\Lambda \cap \Omega^* = \Lambda$, where $\Omega^* = \{\langle x, 1e^{i2\pi}, 0e^{i0} \rangle\}$ represents the universal Complex-HyFS.

This construction preserves logical symmetry. Since the amplitudes r_O and r_P are defined on the unit square, the condition $\min(r_{O_1}, r_{O_2}) \cdot \max(r_{P_1}, r_{P_2}) \leq 1$ holds true, ensuring the intersection is a valid Complex-HyFS.

4.4 Laws and Algebraic Properties

The set-theoretic operations defined for Complex-HyFSs collectively satisfy the fundamental algebraic properties:

1. **Commutativity:**
$$\Lambda_1 \cup \Lambda_2 = \Lambda_2 \cup \Lambda_1, \quad \Lambda_1 \cap \Lambda_2 = \Lambda_2 \cap \Lambda_1.$$
2. **Associativity:**
$$(\Lambda_1 \cup \Lambda_2) \cup \Lambda_3 = \Lambda_1 \cup (\Lambda_2 \cup \Lambda_3), \quad (\Lambda_1 \cap \Lambda_2) \cap \Lambda_3 = \Lambda_1 \cap (\Lambda_2 \cap \Lambda_3).$$
3. **Idempotence:**
$$\Lambda_1 \cup \Lambda_1 = \Lambda_1, \quad \Lambda_1 \cap \Lambda_1 = \Lambda_1.$$
4. **Absorption:**
$$\Lambda_1 \cup (\Lambda_1 \cap \Lambda_2) = \Lambda_1, \quad \Lambda_1 \cap (\Lambda_1 \cup \Lambda_2) = \Lambda_1.$$
5. **Distributivity:**
$$\Lambda_1 \cup (\Lambda_2 \cap \Lambda_3) = (\Lambda_1 \cup \Lambda_2) \cap (\Lambda_1 \cup \Lambda_3),$$
$$\Lambda_1 \cap (\Lambda_2 \cup \Lambda_3) = (\Lambda_1 \cap \Lambda_2) \cup (\Lambda_1 \cap \Lambda_3).$$

These properties confirm that Complex-HyFSs form a consistent lattice structure. The framework extends the cognitive independence of hyperbolic formulations to the complex plane, establishing a robust system for modeling multi-dimensional uncertainty.

5. Complement Operation in Complex-HyFS

In classical fuzzy theory, the complement is defined as $\mu^c = 1 - \mu$, ensuring perfect inversion of membership. However, in many orthopair systems (like CIFS or CPFS), the complement is traditionally achieved by swapping the membership and non-membership degrees.

While swapping is algebraically valid in the Complex-HyFS framework (due to the commutativity of the product $r_O \cdot r_P$), it can lead to semantic degeneracy. For instance, a state of total ignorance represented by $\Lambda(x) = \langle 0, 0 \rangle$ would result in a complement $\Lambda^c(x) = \langle 0, 0 \rangle$ under a swapping regime, which fails to capture the inverse state of "total completeness."

To avoid this and to preserve the cognitive independence of the dimensions, we define the complement by independently inverting the amplitudes and rotating the phases.

Definition 13. *Let Λ be a Complex-HyFS defined on the universe of discourse Ω:*

$$\Lambda = \{\langle x, r_O(x)e^{i\omega_O(x)}, r_P(x)e^{i\omega_P(x)} \rangle : x \in \Omega\}.$$

The complement *of Λ, denoted by Λ^c, is defined as:*

$$\Lambda^c = \left\{ \left\langle x, r_{O^c}(x)e^{i\omega_{O^c}(x)}, r_{P^c}(x)e^{i\omega_{P^c}(x)} \right\rangle : x \in \Omega \right\},$$

where the transformed amplitudes and phases are given by:

$$r_{O^c}(x) = 1 - r_O(x),$$
$$\omega_{O^c}(x) = 2\pi - \omega_O(x),$$
$$r_{P^c}(x) = 1 - r_P(x),$$
$$\omega_{P^c}(x) = 2\pi - \omega_P(x).$$

This definition ensures that the complement operation preserves the **hyperbolic constraint**. Since the original amplitudes $r_O(x)$ and $r_P(x)$ lie within the interval $[0,1]$, their complements $(1-r_O(x))$ and $(1-r_P(x))$ also lie strictly within $[0,1]$. Consequently, their product satisfies the trivial condition:

$$0 \leq (1-r_O(x)) \cdot (1-r_P(x)) \leq 1.$$

Cognitive Interpretation: In this framework, the complement signifies a reversal of attitude in both magnitude and periodicity:

- **Amplitude Reversal:** A high degree of optimism ($r_O \approx 1$) becomes a low degree ($r_{O^c} \approx 0$), capturing the reduction in belief.
- **Phase Rotation:** The phase terms are reflected across the real axis ($2\pi - \omega$), signifying that the "timing" or "context" of the complement is the inverse of the original state.

The corresponding *complex indeterminacy degree* for the complement remains geometrically consistent with the original set, ensuring that Λ^c maintains the structural integrity of the Complex-HyFS family. As

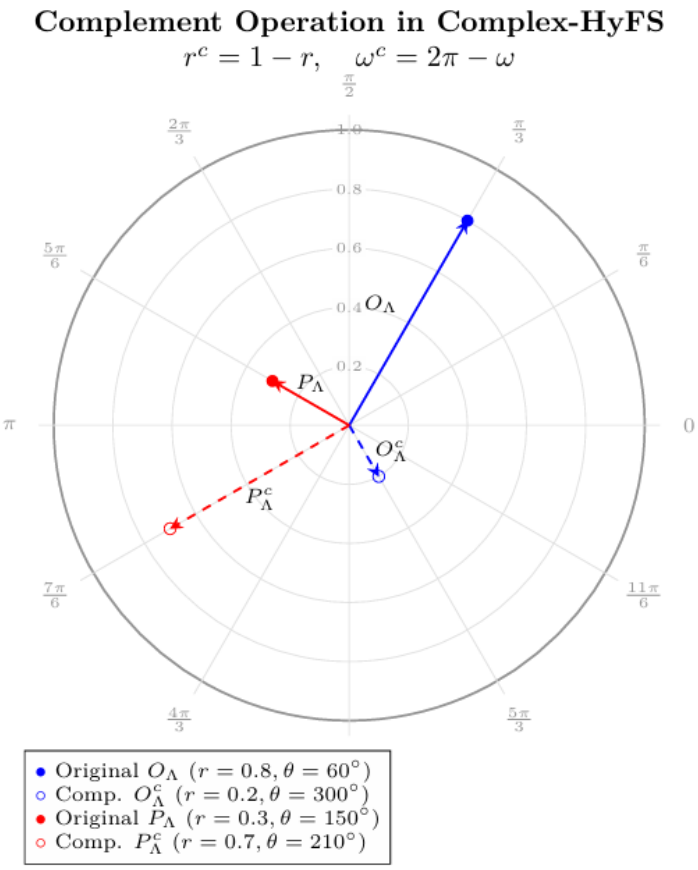

Fig. 4. Complement of a Hyperbolic Fuzzy Set

illustrated in Figure 4, the complement operation in the Complex-HyFS framework is characterized by a dual transformation. The solid vectors denote the original optimistic (O_Λ) and pessimistic (P_Λ) degrees, while the dashed vectors represent their respective complements. This visual representation confirms that the operation preserves the cognitive equilibrium by inverting the amplitude ($1-r$) to reflect the magnitude of negation, while simultaneously rotating the phase by $2\pi - \omega$ to maintain the periodic symmetry inherent in complex fuzzy information.

5.1 Properties of Complement Operators in Complex-HyFSs

The complement operation constitutes a fundamental transformation in the Complex Hyperbolic Fuzzy Set (Complex-HyFS) framework. It preserves *logical symmetry*, *numerical boundedness*, and *cognitive equilibrium* between the optimistic and pessimistic dimensions across both magnitude and phase.

Formally, the mapping

$$\Lambda \mapsto \Lambda^c = \left\{ \left\langle x, (1-r_O(x))e^{i(2\pi-\omega_O(x))}, (1-r_P(x))e^{i(2\pi-\omega_P(x))} \right\rangle : x \in \Omega \right\}$$

is designed to remain valid under the hyperbolic constraint

$$r_{O_{\Lambda^c}}(x) \cdot r_{P_{\Lambda^c}}(x) \leq 1,$$

ensuring that the complement Λ^c inherits the same admissible domain as Λ.
This formulation allows the complementary pair to retain cognitive independence while maintaining the structural harmony of the complex hybrid topology. The following theorems establish the principal algebraic and logical properties.

Theorem 11. (Involution Property) *For every Complex-HyFS Λ on Ω, the complement is an involution, that is,*

$$(\Lambda^c)^c = \Lambda.$$

Proof. Applying the definition twice to the amplitude and phase components:

$$r_{(O^c)^c}(x) = 1 - (1 - r_O(x)) = r_O(x),$$
$$\omega_{(O^c)^c}(x) = 2\pi - (2\pi - \omega_O(x)) = \omega_O(x).$$

Similarly, for the pessimistic component:

$$r_{(P^c)^c}(x) = 1 - (1 - r_P(x)) = r_P(x),$$
$$\omega_{(P^c)^c}(x) = 2\pi - (2\pi - \omega_P(x)) = \omega_P(x).$$

This restores the original complex degrees. Therefore, $(\Lambda^c)^c = \Lambda$. ■

Theorem 12. (Monotonicity Property) *Let Λ_1 and Λ_2 be two Complex-HyFSs such that $\Lambda_1 \subseteq \Lambda_2$, defined by amplitude dominance:*

$$r_{O_1}(x) \leq r_{O_2}(x) \quad \text{and} \quad r_{P_1}(x) \geq r_{P_2}(x).$$

Then, their complements satisfy the reverse inclusion:

$$\Lambda_1^c \supseteq \Lambda_2^c.$$

Proof. From the complement definition:

$$r_{O_1^c}(x) = 1 - r_{O_1}(x) \geq 1 - r_{O_2}(x) = r_{O_2^c}(x),$$

and

$$r_{P_1^c}(x) = 1 - r_{P_1}(x) \leq 1 - r_{P_2}(x) = r_{P_2^c}(x).$$

Hence, for all $x \in \Omega$, the pairwise relations hold, establishing $\Lambda_1^c \supseteq \Lambda_2^c$. ■

Theorem 13. (Boundedness and Normalization) *For all $x \in \Omega$, if Λ is a Complex-HyFS, then its complement Λ^c also satisfies the boundary conditions:*

$$0 \leq r_{\Lambda^c}(x) \leq 1, \quad 0 \leq \omega_{\Lambda^c}(x) \leq 2\pi,$$

and the hyperbolic constraint

$$r_{O_{\Lambda^c}}(x) \cdot r_{P_{\Lambda^c}}(x) \leq 1.$$

Proof. Amplitudes: Since $r_O, r_P \in [0,1]$, their complements $1 - r_O$ and $1 - r_P$ also belong to $[0,1]$. The product of any two numbers in the unit interval is at most 1, satisfying the hyperbolic constraint. **Phases:** Since $\omega \in [0, 2\pi]$, the reflection $2\pi - \omega$ remains within $[0, 2\pi]$. Hence, Λ^c remains a valid Complex-HyFS. ■

Theorem 14. (Soft Complement Identity Laws in Complex-HyFSs) *For any Complex-HyFS Λ, the following soft inclusions hold for all $x \in \Omega$:*

$$\emptyset^* \subseteq \Lambda \cap \Lambda^c \subseteq \Lambda \cup \Lambda^c \subseteq \Omega^*,$$

where the complex universal and null sets are defined as:

$$\Omega^* = \{\langle x, 1e^{i2\pi}, 0e^{i0}\rangle\} \quad \textit{and} \quad \emptyset^* = \{\langle x, 0e^{i0}, 1e^{i2\pi}\rangle\}.$$

Proof. Considering the amplitudes for the union $\Lambda \cup \Lambda^c$:

$$r_{O_\cup}(x) = \max(r_O(x), 1 - r_O(x)) \leq 1, \quad r_{P_\cup}(x) = \min(r_P(x), 1 - r_P(x)) \geq 0.$$

Hence, $\Lambda \cup \Lambda^c \subseteq \Omega^*$.
Similarly, for the intersection $\Lambda \cap \Lambda^c$:

$$r_{O_\cap}(x) = \min(r_O(x), 1 - r_O(x)) \geq 0, \quad r_{P_\cap}(x) = \max(r_P(x), 1 - r_P(x)) \leq 1,$$

so $\emptyset^* \subseteq \Lambda \cap \Lambda^c$.
These relations capture the *soft inclusion* principle: the union and intersection of a complex fuzzy set with its complement do not yield absolute universal or null sets, but rather graded complex approximations. ■

Example 1. (Failure of Classical Laws in Complex-HyFS) *Let a Complex-HyFS element be defined as $\Lambda(x) = \langle 0.4e^{i0.5\pi}, 0.3e^{i\pi}\rangle$. Then, the complement is calculated as:*

$$\Lambda^c(x) = \langle 0.6e^{i1.5\pi}, 0.7e^{i\pi}\rangle.$$

The union $\Lambda \cup \Lambda^c$ (applying Max for Optimism and Min for Pessimism) results in:

$$\Lambda \cup \Lambda^c = \langle 0.6e^{i1.5\pi}, 0.3e^{i\pi}\rangle \neq \langle 1e^{i2\pi}, 0e^{i0}\rangle.$$

This inequality demonstrates that the classical Law of Excluded Middle does not hold strictly in the Complex-HyFS framework, signifying a state of cognitive coexistence rather than binary exclusivity.

Theorem 15. (Complement Symmetry and Cognitive Balance) *The complement operation in Complex-HyFSs preserves* cognitive balance *between the optimistic and pessimistic components. For every element $x \in \Omega$:*

1. ***Amplitude Sum:*** $r_{O_\Lambda}(x) + r_{O_{\Lambda^c}}(x) = 1$ *and* $r_{P_\Lambda}(x) + r_{P_{\Lambda^c}}(x) = 1$.
2. ***Phase Sum:*** $\omega_{O_\Lambda}(x) + \omega_{O_{\Lambda^c}}(x) = 2\pi$ *and* $\omega_{P_\Lambda}(x) + \omega_{P_{\Lambda^c}}(x) = 2\pi$.

Proof. Substituting the definitions directly yields the results. The phase sum equal to 2π confirms that the complement completes the periodic cycle of the original set, ensuring a holistic interpretation of complex uncertainty. ■

5.2 De Morgan's Laws for the Complement Operation

The complement operation in the Complex-HyFS framework follows a symmetric and logically consistent transformation between the optimistic and pessimistic components. Since Complex-HyFS involves both magnitude (amplitude) and periodicity (phase), it is necessary to verify that classical logical relationships remain valid for both dimensions.
Let Λ_1 and Λ_2 be two Complex-HyFSs defined on the same universe Ω. Then, the following *De Morgan's Laws* hold for the complement operation.

Theorem 16. (De Morgan's Laws for Complex-HyFSs) *For any two Complex Hyperbolic Fuzzy Sets Λ_1 and Λ_2 on Ω,*

$$(\Lambda_1 \cup \Lambda_2)^c = \Lambda_1^c \cap \Lambda_2^c, \quad \textit{and} \quad (\Lambda_1 \cap \Lambda_2)^c = \Lambda_1^c \cup \Lambda_2^c.$$

Proof. Let $x \in \Omega$. We represent the sets in polar coordinates:

$$\Lambda_k = \{\langle r_{O_k}(x)e^{i\omega_{O_k}(x)}, r_{P_k}(x)e^{i\omega_{P_k}(x)}\rangle\}, \quad k = 1, 2.$$

Part 1: Proof for Amplitudes
Recall that the complement of amplitude is defined as $r^c = 1 - r$. For the union $\Lambda_1 \cup \Lambda_2$, the optimistic amplitude is $r_{O_\cup} = \max\{r_{O_1}, r_{O_2}\}$. The amplitude of the complement of the union is:

$$r_{O_{(\cup)^c}} = 1 - \max\{r_{O_1}, r_{O_2}\} = \min\{1 - r_{O_1}, 1 - r_{O_2}\}.$$

Now consider the intersection of complements $\Lambda_1^c \cap \Lambda_2^c$. The intersection operation takes the minimum of the optimistic components:

$$r_{O_{\cap^c}} = \min\{r_{O_1^c}, r_{O_2^c}\} = \min\{1 - r_{O_1}, 1 - r_{O_2}\}.$$

Thus, the amplitude components match. The same logic applies to the pessimistic amplitudes (where union uses $\min$ and intersection uses $\max$), as $1 - \min(a, b) = \max(1 - a, 1 - b)$.
Part 2: Proof for Phases
Recall that the complement of phase is defined as $\omega^c = 2\pi - \omega$. For the union $\Lambda_1 \cup \Lambda_2$, the optimistic phase is $\omega_{O_\cup} = \max\{\omega_{O_1}, \omega_{O_2}\}$. The phase of the complement of the union is:

$$\omega_{O_{(\cup)^c}} = 2\pi - \max\{\omega_{O_1}, \omega_{O_2}\}.$$

Using the algebraic property that $C - \max(A, B) = \min(C - A, C - B)$, we obtain:

$$\omega_{O_{(\cup)^c}} = \min\{2\pi - \omega_{O_1}, 2\pi - \omega_{O_2}\}.$$

Now consider the intersection of complements $\Lambda_1^c \cap \Lambda_2^c$. The intersection takes the minimum of the optimistic phases:

$$\omega_{O_{\cap^c}} = \min\{\omega_{O_1^c}, \omega_{O_2^c}\} = \min\{2\pi - \omega_{O_1}, 2\pi - \omega_{O_2}\}.$$

The phase components match perfectly.

Since both amplitude and phase components satisfy the equality for O and P, we conclude that $(\Lambda_1 \cup \Lambda_2)^c = \Lambda_1^c \cap \Lambda_2^c$. A similar argument applies to the second law $(\Lambda_1 \cap \Lambda_2)^c = \Lambda_1^c \cup \Lambda_2^c$. ■

Remark 2. *De Morgan's Laws guarantee algebraic consistency and logical symmetry within the Complex-HyFS framework. They confirm that the complement operation, defined by the transformation* $(1-r, 2\pi-\omega)$, *exhibits behavior analogous to classical fuzzy complements while strictly preserving the hyperbolic constraint* ($r_O \cdot r_P \leq 1$). *This ensures that the algebraic structure of Complex-HyFSs remains both self-consistent and logically reversible in the complex plane.*

Theorem 17. (Extended De Morgan's Laws) *Let* $\{\Lambda_i\}_{i=1}^N$ *be a finite family of Complex-HyFSs on the universe* Ω. *Then, the complement operator distributes over finite unions and intersections as follows:*

$$\left(\bigcup_{i=1}^N \Lambda_i\right)^c = \bigcap_{i=1}^N \Lambda_i^c, \qquad \left(\bigcap_{i=1}^N \Lambda_i\right)^c = \bigcup_{i=1}^N \Lambda_i^c.$$

Proof. The result follows by mathematical induction on N. For the base case $N = 2$, the property has already been established in the previous theorem (De Morgan's Laws for two sets).

Assume the property holds for $N = k$. Then, for $N = k + 1$:

$$\left(\bigcup_{i=1}^{k+1} \Lambda_i\right)^c = \left(\left(\bigcup_{i=1}^{k} \Lambda_i\right) \cup \Lambda_{k+1}\right)^c.$$

By applying the De Morgan law for two sets (treating $\cup_{i=1}^k \Lambda_i$ as a single set), we obtain:

$$= \left(\bigcup_{i=1}^{k} \Lambda_i\right)^c \cap \Lambda_{k+1}^c.$$

By the inductive hypothesis, $\left(\bigcup_{i=1}^{k} \Lambda_i\right)^c = \bigcap_{i=1}^{k} \Lambda_i^c$. Substituting this back yields:

$$= \left(\bigcap_{i=1}^{k} \Lambda_i^c\right) \cap \Lambda_{k+1}^c = \bigcap_{i=1}^{k+1} \Lambda_i^c.$$

The corresponding intersection case follows analogously by duality:

$$\left(\bigcap_{i=1}^{k+1} \Lambda_i\right)^c = \left(\left(\bigcap_{i=1}^{k} \Lambda_i\right) \cap \Lambda_{k+1}\right)^c = \left(\bigcap_{i=1}^{k} \Lambda_i\right)^c \cup \Lambda_{k+1}^c = \bigcup_{i=1}^{k+1} \Lambda_i^c.$$

Thus, the complement operator satisfies extended De Morgan's Laws for any finite collection of Complex-HyFSs. This implies that the properties hold for both the amplitude terms (preserving the hyperbolic constraint) and the phase terms (preserving periodic symmetry). ■

Theorem 18. (Triple and Nested De Morgan Property in Complex-HyFSs) *For any three Complex Hyperbolic Fuzzy Sets Λ_1, Λ_2, and Λ_3 defined on the same universe Ω, the following extended De Morgan identities hold:*

$$(\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)^c = \Lambda_1^c \cap \Lambda_2^c \cap \Lambda_3^c, \qquad (\Lambda_1 \cap \Lambda_2 \cap \Lambda_3)^c = \Lambda_1^c \cup \Lambda_2^c \cup \Lambda_3^c.$$

Furthermore, for any nested combination of union and intersection, the complement satisfies:

$$(\Lambda_1 \cup (\Lambda_2 \cap \Lambda_3))^c = \Lambda_1^c \cap (\Lambda_2^c \cup \Lambda_3^c), \qquad (\Lambda_1 \cap (\Lambda_2 \cup \Lambda_3))^c = \Lambda_1^c \cup (\Lambda_2^c \cap \Lambda_3^c).$$

Proof. Using the De Morgan identities for two Complex-HyFSs:

$$(\Lambda_i \cup \Lambda_j)^c = \Lambda_i^c \cap \Lambda_j^c, \qquad (\Lambda_i \cap \Lambda_j)^c = \Lambda_i^c \cup \Lambda_j^c,$$

we extend the result via algebraic manipulation.

For the triple case:

$$(\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)^c = ((\Lambda_1 \cup \Lambda_2) \cup \Lambda_3)^c = (\Lambda_1 \cup \Lambda_2)^c \cap \Lambda_3^c = (\Lambda_1^c \cap \Lambda_2^c) \cap \Lambda_3^c = \Lambda_1^c \cap \Lambda_2^c \cap \Lambda_3^c.$$

The dual expression follows by symmetry:

$$(\Lambda_1 \cap \Lambda_2 \cap \Lambda_3)^c = ((\Lambda_1 \cap \Lambda_2) \cap \Lambda_3)^c = (\Lambda_1 \cap \Lambda_2)^c \cup \Lambda_3^c = (\Lambda_1^c \cup \Lambda_2^c) \cup \Lambda_3^c = \Lambda_1^c \cup \Lambda_2^c \cup \Lambda_3^c.$$

For the nested form, applying the complement to the outer operation first:

$$(\Lambda_1 \cup (\Lambda_2 \cap \Lambda_3))^c = \Lambda_1^c \cap (\Lambda_2 \cap \Lambda_3)^c.$$

Applying De Morgan's law to the inner term $(\Lambda_2 \cap \Lambda_3)^c$:

$$= \Lambda_1^c \cap (\Lambda_2^c \cup \Lambda_3^c).$$

By duality for the second nested case:

$$(\Lambda_1 \cap (\Lambda_2 \cup \Lambda_3))^c = \Lambda_1^c \cup (\Lambda_2 \cup \Lambda_3)^c = \Lambda_1^c \cup (\Lambda_2^c \cap \Lambda_3^c).$$

Thus, all extended and nested De Morgan laws are valid for Complex-HyFSs, ensuring algebraic closure and logical harmony under complement operations within the complex hybrid fuzzy framework. ■

Proposition 19. (Dual Inclusion under Complement) *Let Λ_1 and Λ_2 be two Complex-HyFSs defined on the same universe Ω. If $\Lambda_1 \subseteq \Lambda_2$, then the complement operator reverses the inclusion order, that is,*

$$\Lambda_1^c \supseteq \Lambda_2^c.$$

Proof. For every $x \in \Omega$, the inclusion $\Lambda_1 \subseteq \Lambda_2$ is defined by the dominance of their amplitude terms:

$$r_{O_1}(x) \leq r_{O_2}(x), \qquad r_{P_1}(x) \geq r_{P_2}(x).$$

Applying the complement transformation defined for Complex-HyFS (where $r^c = 1 - r$), we obtain the amplitudes for the complements:

$$r_{O_1^c}(x) = 1 - r_{O_1}(x),$$
$$r_{O_2^c}(x) = 1 - r_{O_2}(x).$$

Since $r_{O_1}(x) \leq r_{O_2}(x)$, it follows algebraically that $1 - r_{O_1}(x) \geq 1 - r_{O_2}(x)$, and thus:

$$r_{O_1^c}(x) \geq r_{O_2^c}(x).$$

Similarly, for the pessimistic components:

$$r_{P_1^c}(x) = 1 - r_{P_1}(x) \leq 1 - r_{P_2}(x) = r_{P_2^c}(x).$$

Hence, the conditions for inclusion are reversed (O^c is greater, P^c is smaller). By the definition of set inclusion for fuzzy structures, this implies $\Lambda_1^c \supseteq \Lambda_2^c$. This confirms the monotone decreasing property of the complement operator in the Complex-HyFS framework. ■

Theorem 20. (Inclusion–De Morgan Compatibility) *For any two Complex-HyFSs Λ_1 and Λ_2 defined on the same universe Ω, the inclusion relation and the complement operation are dual under De Morgan's framework, that is,*

$$\Lambda_1 \subseteq \Lambda_2 \iff \Lambda_1^c \supseteq \Lambda_2^c.$$

Proof. The forward implication (⇒) follows directly from the monotonicity of the complement operator established in the previous proposition.
For the converse implication (⇐), we assume $\Lambda_1^c \supseteq \Lambda_2^c$. Applying the complement operation a second time involves:

- **Amplitudes:** $r^{(c)^c} = 1 - (1 - r) = r$.
- **Phases:** $\omega^{(c)^c} = 2\pi - (2\pi - \omega) = \omega$ (modulo 2π).

Using the involution property $(\Lambda^c)^c = \Lambda$, we derive:

$$\Lambda_1^c \supseteq \Lambda_2^c \Rightarrow (\Lambda_1^c)^c \subseteq (\Lambda_2^c)^c \Rightarrow \Lambda_1 \subseteq \Lambda_2.$$

Therefore, the inclusion and complement operations exhibit perfect duality in the Complex-HyFS structure, confirming their compatibility with De Morgan's laws. ■

Theorem 21. (Hyperbolic Consistency of De Morgan's Laws) *Let Λ_1 and Λ_2 be two Complex Hyperbolic Fuzzy Sets defined on the same universe Ω, satisfying the hyperbolic constraint for their amplitudes:*

$$r_{O_{\Lambda_i}}(x) \cdot r_{P_{\Lambda_i}}(x) \leq 1, \quad i = 1, 2.$$

Then, their De Morgan complements $(\Lambda_1 \cup \Lambda_2)^c$ and $(\Lambda_1 \cap \Lambda_2)^c$ also satisfy the same constraint, i.e.,

$$r_{O_{(\Lambda_1 \cup \Lambda_2)^c}}(x) \cdot r_{P_{(\Lambda_1 \cup \Lambda_2)^c}}(x) \leq 1,$$

and similarly for $(\Lambda_1 \cap \Lambda_2)^c$.

Proof. The complement operation in Complex-HyFSs transforms the amplitudes (r_O, r_P) into $(1-r_O, 1-r_P)$.

Consider the De Morgan complement of the union $(\Lambda_1 \cup \Lambda_2)^c$. The amplitudes are given by:

$$r_{O_{(\cup)^c}} = \min(1 - r_{O_1}, 1 - r_{O_2}), \quad r_{P_{(\cup)^c}} = \max(1 - r_{P_1}, 1 - r_{P_2}).$$

Since the original amplitudes are normalized in the unit interval $[0, 1]$, their complements $1 - r$ are also strictly within $[0, 1]$. The product of any two real numbers strictly bounded within the unit interval $[0, 1]$ is necessarily less than or equal to 1.

Mathematically:

$$0 \leq r_{O_{(\cup)^c}}(x) \leq 1 \quad \text{and} \quad 0 \leq r_{P_{(\cup)^c}}(x) \leq 1 \implies r_{O_{(\cup)^c}}(x) \cdot r_{P_{(\cup)^c}}(x) \leq 1.$$

Therefore, the complement of any union or intersection of Complex-HyFSs remains within the same hyperbolic feasible region (the unit square). This confirms the structural and algebraic consistency of De Morgan's laws in this extended complex fuzzy framework. ■

6. Construction of Complex-HyFS from Classical Complex Fuzzy Sets

The Complex Hyperbolic Fuzzy Set (Complex-HyFS) can be systematically constructed from a standard Complex Fuzzy Set (CFS) by introducing appropriate complement operators that generate the pessimistic component corresponding to each optimistic (membership) degree. This construction provides a coherent framework connecting Ramot's complex fuzzy concept to the dual-component hyperbolic system.
Let Ω be a universal set. A standard Complex Fuzzy Set Λ_{CFS} is characterized by a single complex membership function:

$$O_\Lambda(x) = r_O(x) \cdot e^{i\omega_O(x)},$$

where $r_O(x) \in [0, 1]$ represents the amplitude of belongingness and $\omega_O(x)$ represents the phase (periodicity). This single-valued formulation expresses only the extent of acceptance, leaving the independent notion of rejection undefined.
To generalize this into the Complex-HyFS structure, a *complex pessimistic degree* $P_\Lambda(x) = r_P(x) \cdot e^{i\omega_P(x)}$ is generated using a generator tuple $\Psi = (\varphi, \theta)$ acting on the amplitude and phase respectively:

$$\begin{aligned} r_P(x) &= \varphi(r_O(x)), \\ \omega_P(x) &= \theta(\omega_O(x)). \end{aligned}$$

Here, $\varphi : [0, 1] \to [0, 1]$ is a fuzzy negation operator (monotone decreasing, $\varphi(0) = 1, \varphi(1) = 0$), and θ is a phase transformation function (typically $\theta(\omega) = 2\pi - \omega$).
The resulting constructed Complex-HyFS is:

$$\Lambda = \left\{ \left\langle x, r_O(x)e^{i\omega_O(x)}, \varphi(r_O(x))e^{i\theta(\omega_O(x))} \right\rangle : x \in \Omega \right\}.$$

Due to the hyperbolic condition ($r_O \cdot r_P \leq 1$), any standard fuzzy complement φ ensures validity, as the product of two numbers in $[0, 1]$ is always ≤ 1.

Example 2. (Complement-Based Construction of Complex-HyFS) *We illustrate the construction of Complex-HyFS using two different types of complement operators for the amplitude, combined with a standard phase rotation.*

1. ***Standard (Zadeh) Construction:***
   Let the amplitude complement be the standard negation $\varphi(t) = 1 - t$ and the phase rotation be $\theta(\omega) = 2\pi - \omega$.
   Consider an element x with an optimistic degree $O_\Lambda(x) = 0.7e^{i0.5\pi}$. The pessimistic components are generated as follows:

   - ***Amplitude:*** $r_P(x) = 1 - 0.7 = 0.3$.

- ***Phase:*** $\omega_P(x) = 2\pi - 0.5\pi = 1.5\pi$.

Check Constraint:

$$r_O(x) \cdot r_P(x) = 0.7 \times 0.3 = 0.21 \leq 1.$$

Since the product is less than or equal to 1, the construction is valid.
Result: $\Lambda(x) = \langle 0.7e^{i0.5\pi}, 0.3e^{i1.5\pi} \rangle$.

2. ***Yager Construction:***
Let the amplitude complement be defined by Yager's generator $\varphi(t) = (1-t^2)^{1/2}$ *and the phase rotation be* $\theta(\omega) = 2\pi - \omega$.
Consider an element with $O_\Lambda(x) = 0.6e^{i\pi}$.

- ***Amplitude:*** $r_P(x) = \sqrt{1-(0.6)^2} = \sqrt{0.64} = 0.8$.
- ***Phase:*** $\omega_P(x) = 2\pi - \pi = \pi$.

Check Constraint:

$$r_O(x) \cdot r_P(x) = 0.6 \times 0.8 = 0.48 \leq 1.$$

Note: *In Pythagorean sets,* $0.6^2 + 0.8^2 = 1$ *(boundary case), but in Complex-HyFS, the product* 0.48 *lies well within the interior of the feasible unit square.*
Result: $\Lambda(x) = \langle 0.6e^{i\pi}, 0.8e^{i\pi} \rangle$.

7. Complement Operation in Complex-HyFS via Fuzzy Operators

The complement operation plays a vital role in extending complex fuzzy principles. Within the Complex-HyFS environment, complements can be systematically constructed using mixed-type operators for the amplitudes and rotational operators for the phases.

Definition 14. *Let* φ_1 *and* φ_2 *be two fuzzy complement operators acting on the amplitudes, and* θ *be a phase rotation operator. The* generalized complement *of a Complex-HyFS* Λ *is defined as:*

$$\Lambda^c = \left\{ \left\langle x, \varphi_1(r_O(x))e^{i\theta(\omega_O(x))}, \varphi_2(r_P(x))e^{i\theta(\omega_P(x))} \right\rangle : x \in \Omega \right\}.$$

Common amplitude operators include:

- **Standard:** $\varphi(t) = 1 - t$,
- **Sugeno:** $\varphi(t) = \frac{1-t}{1+\lambda t}$, $\lambda > -1$.

The standard phase operator is $\theta(\omega) = 2\pi - \omega$ (or $2\pi - \omega \pmod{2\pi}$).

Theorem 22. (Universal Closure under Complement in Complex-HyFS) *Let* Λ *be a Complex Hyperbolic Fuzzy Set. Then, its complement* Λ^c *constructed using any pair of standard fuzzy negations* $\varphi_1, \varphi_2 : [0,1] \to [0,1]$ *is always a valid Complex-HyFS.*

Proof. The validity of a Complex-HyFS depends on the hyperbolic condition:

$$r_{new\_O} \cdot r_{new\_P} \leq 1.$$

By definition, fuzzy complement operators map values from $[0,1]$ to $[0,1]$. Let $A = \varphi_1(r_O(x))$ and $B = \varphi_2(r_P(x))$. Since $A \in [0,1]$ and $B \in [0,1]$, their product must satisfy:

$$A \cdot B \leq 1 \cdot 1 = 1.$$

Thus, unlike Pythagorean or q-rung orthopair sets which require specific parameter checks (e.g., $a^2+b^2 \leq 1$), the Complex-HyFS framework guarantees closure for any valid amplitude mapping within the unit square. ■
Table **??** presents a summary of operators applicable to the amplitudes of Complex-HyFS.

Table 1
Typical fuzzy complement operators applicable to Complex-HyFS amplitudes.

Type	Amplitude Function $\varphi(r)$	Complex-HyFS Validity
Standard	$1 - r$	Always Valid ($r_O \cdot r_P \leq 1$)
Yager	$(1 - r^p)^{1/p}, p > 0$	Always Valid
Hamacher	$\frac{1-r}{1+\lambda r(1-r)}$	Always Valid
Sugeno	$\frac{1-r}{1+\alpha r}$	Always Valid

Example 3. (Mixed-Type Complement Construction) *Let a complex fuzzy element be defined as* $\Lambda(x) = \langle 0.9e^{i0.2\pi}, 0.5e^{i1.2\pi} \rangle$.
Note that this point is valid in the Complex-HyFS framework ($0.9 \times 0.5 = 0.45 \leq 1$) *but would be invalid in Complex Pythagorean Fuzzy Sets* ($0.9^2 + 0.5^2 = 0.81 + 0.25 = 1.06 > 1$).
We construct a mixed-type complement component-wise by applying a Standard complement to the original Optimism and a Sugeno complement (with parameter $\alpha = 1$*) to the original Pessimism:*

1. ***New Optimism (from old O):***
   Using the Standard operator on $r_O = 0.9$*:*
   $$r_{O^c} = 1 - 0.9 = 0.1.$$
   Phase transformation: $2\pi - 0.2\pi = 1.8\pi$.
2. ***New Pessimism (from old P):***
   Using the Sugeno operator on $r_P = 0.5$*:*
   $$r_{P^c} = \frac{1 - 0.5}{1 + 1(0.5)} = \frac{0.5}{1.5} \approx 0.33.$$
   Phase transformation: $2\pi - 1.2\pi = 0.8\pi$.

Result: $\Lambda^c = \langle 0.1e^{i1.8\pi}, 0.33e^{i0.8\pi} \rangle$.
Check: $0.1 \times 0.33 = 0.033 \leq 1$. *The resulting set remains a valid Complex-HyFS.*

8. Applications and Future Scope

Existing extensions of fuzzy set theory have demonstrated remarkable applicability across diverse multi-criteria decision-making (MCDM) domains. Intuitionistic and Pythagorean fuzzy models have contributed significantly to organizational evaluation and spatial reasoning. However, these models are often limited by restrictive boundary conditions (triangular or circular) and a lack of periodicity.
The proposed Complex Hyperbolic Fuzzy Set (Complex-HyFS) generalizes these models by introducing two distinct mechanisms. First, it employs a Hyperbolic Interaction defined by the constraint $r_O \cdot r_P \leq 1$, which covers the full unit square, thereby allowing for high simultaneous acceptance and rejection. Second, it integrates a Complex Phase as a second dimension to model periodicity, seasonality, or directional data.
In future research, Complex-HyFSs can be effectively applied to areas requiring asymmetric reasoning and periodic uncertainty handling. One prominent application is Bio-Signal Analysis, where the framework can model periodic medical data (such as ECG/EEG) with the amplitude representing anomaly severity and the phase representing the specific time-point in the cardiac or neural cycle. Additionally, in Seasonal Economic Forecasting, the model allows for the quantification of investor optimism (O) versus pessimism (P) where the phase captures the fiscal quarter or seasonal market trends. Furthermore, the framework is applicable to Periodic Cybersecurity Assessment for representing system vulnerability where attack likelihoods (O) and defense capabilities (P) vary periodically with server load cycles. Finally, Interference Modeling can utilize the wave-like properties of the complex phase to model constructive

and destructive interference in decision-making consensus.
It is observed that the Complex-HyFS establishes a promising and versatile foundation for hybrid uncertainty modeling, uniting the algebraic freedom of hyperbolic geometry with the temporal richness of complex analysis.

9. Conclusion

The evolution of fuzzy set theory has consistently aimed to expand the mathematical space available for modeling uncertainty, progressing from standard sets to increasingly flexible orthopair structures. In this paper, we introduced the Complex Hyperbolic Fuzzy Set (Complex-HyFS), a novel framework that synergizes the algebraic flexibility of hyperbolic geometry with the vector-based richness of complex analysis. Unlike conventional Intuitionistic or Pythagorean models, which are bound by linear or quadratic constraints, the Complex-HyFS adopts the hyperbolic condition $r_O(x) \cdot r_P(x) \leq 1$. This geometric reinterpretation effectively unlocks the entire unit square in the amplitude plane, resolving the limitations of restrictive boundary conditions and enabling the accurate representation of ambivalent decision states characterized by high simultaneous optimism and pessimism.
From a theoretical perspective, the Complex-HyFS framework establishes a robust algebraic structure that is closed under fundamental set-theoretic operations. We have rigorously demonstrated that the framework satisfies the Extended De Morgan's Laws and that the complement operator preserves the hyperbolic constraint without degeneracy. Crucially, the integration of the complex phase term transforms static evaluations into dynamic, periodic assessments. By capturing not just the magnitude of belief and doubt, but also their temporal context and directionality, the model significantly enhances the representational power of fuzzy systems for dynamic phenomena.
In practical terms, the proposed framework offers a versatile tool for advanced decision-making in complex environments. By decoupling the magnitude of judgment from restrictive constraints and adding a dimension for periodicity, Complex-HyFS is particularly well-suited for applications involving seasonality and cyclic data, such as bio-signal processing, periodic cybersecurity assessment, and economic forecasting. Future research will focus on developing specialized aggregation operators, correlation coefficients, and distance measures specific to Complex-HyFS to facilitate its direct implementation in multi-criteria decision-making algorithms and intelligent support systems.

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Conflicts of Interest

The authors declare no conflicts of interest.

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