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# Circular Fractional Orthopair Fuzzy MULTIMOORA approach with Schweizer-Sklar Aggregation for Cloud Computing Service Provider Selection

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**ABSTRACT**

Selecting a suitable cloud computing service provider among several competing vendors is a demanding multi-attribute decision-making problem because the alternatives must be weighed against criteria such as security, scalability, cost-effectiveness, and service reliability, while the judgments supplied by decision makers are seldom crisp or complete. This study extends the fractional orthopair fuzzy set to the circular fractional orthopair fuzzy set by replacing the q-rung orthopair fuzzy set with the circular q-rung orthopair fuzzy set, since the latter incorporates a radius term and therefore covers a disk-shaped information region instead of a single point. Some basic mathematical concepts associated with the proposed set, including its score and accuracy functions and fundamental operational laws, are introduced. Furthermore, Schweizer-Sklar operations are formulated within this new environment, and a family of weighted averaging and geometric aggregation operators is studied along with their related properties. Based on the proposed theory, the multi-objective optimization by ratio analysis plus full multiplicative form (MULTIMOORA) method is developed and applied to the selection of a suitable cloud computing service provider, followed by sensitivity and comparative analyses.

## 1. Introduction

Cloud computing has become a defining feature of modern digital infrastructure, offering organizations on-demand access to a shared pool of configurable computing resources such as servers, storage, and software, without the burden of maintaining these resources in-house [1]. This model has been adopted across banking, healthcare, education, and government services, since it allows enterprises to scale their operations rapidly while reducing capital expenditure on physical hardware. As the number of vendors offering these services has grown, the decision of which provider to entrust with an organization's data and workloads has itself become a strategic concern that directly affects an organization's performance, security, and cost-effectiveness [2].

Choosing a suitable cloud service provider from among several competing vendors is a genuine multi-attribute decision-making problem, since providers must be compared against numerous conflicting criteria such as security guarantees, scalability, service-level reliability, pricing structure, and support quality. These criteria are rarely quantifiable in a precise manner, and the opinions gathered from organizational stakeholders during vendor evaluation are typically vague, incomplete, or expressed in imprecise linguistic terms rather than crisp numbers. Several hybrid decision-making frameworks have already been proposed to cope with this uncertainty, ranging from a ranking framework built around measurable quality-of-service attributes [3] to a hybrid multi-criteria model that compares services on cost and

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performance grounds [4], and more recently to a pentagonal intuitionistic fuzzy AHP-TOPSIS approach designed specifically for the cloud service provider selection problem [5].

Building on this line of work, interval-valued q-rung orthopair fuzzy soft sets have recently been combined with a correlation-based TOPSIS procedure to resolve an optimal cloud service provider selection problem, illustrating that richer fuzzy representations can meaningfully improve the discrimination among competing providers [6]. However, none of the existing frameworks for this application has yet exploited a circular representation, which covers a disk-shaped region of admissible values rather than a single point, nor a fractional-order membership constraint, which is better suited to capturing extreme or borderline expert judgments. Motivated by this gap, the present study develops a circular fractional orthopair fuzzy decision-making framework equipped with Schweizer-Sklar aggregation operators and applies it to the selection of a suitable cloud computing service provider, supported by sensitivity and comparative analysis.

### 1.1 Overview of Fuzzy Modeling Approaches

Multi-attribute decision-making (MADM) rarely reduces to a comparison of crisp numbers, since expert judgments are almost always coloured by vagueness and disagreement. Zadeh's fuzzy set (FS) addressed this by letting an element belong to a set by degree, via a membership function taking values in $[0, 1]$ [7]. However, it offered no independent channel for rejection, so two experts agreeing on the strength of support but disagreeing on the strength of opposition were indistinguishable under a FS. Atanassov closed this gap with the intuitionistic fuzzy set (IFS), attaching a non-membership grade $N$ alongside the membership grade $M$, subject to the linear bound $M + N \leq 1$, with hesitancy given by the residual $1 - M - N$ [8]. Its principal weakness is that whenever an expert wishes to assign comparatively high values to both $M$ and $N$ simultaneously, the pair falls outside the admissible region, forcing an artificial deflation of one grade. Yager relaxed this restriction by replacing the linear bound with a quadratic one, $M^2 + N^2 \leq 1$, giving rise to the Pythagorean fuzzy set (PFS) and enlarging the feasible region [9]. Since no single fixed exponent can suit every decision context, Yager generalised further through the $q$-rung orthopair fuzzy set (q-ROFS), imposing $M^q + N^q \leq 1, \quad q \geq 1,\ q \in \mathbb{Z},$ [10]. As $q$ increases, the admissible region expands monotonically, so the q-ROFS nests the IFS ($q = 1$) and the PFS ($q = 2$) as special cases, giving the decision-maker direct control over how permissive the model should be.

A separate limitation persists in the q-ROFS: the same exponent $q$ governs both grades at once, so an expert confident about acceptance but unsure about rejection has no way to express that asymmetry, since raising $q$ loosens both sides of the constraint together. Seikh and Mandal addressed this by decoupling the exponents, proposing the $(p, q)$-quasirung orthopair fuzzy set (pq-QOFS) under $M^p + N^q \leq 1$, where $p$ and $q$ need not coincide [11]. This was needed because a single shared rung could not scale membership and non-membership at different rates. Yet $p$ and $q$ remain confined to positive integers in this formulation, so an expert whose confidence sits between two integer levels, for instance midway between $q = 1$ and $q = 2$, still cannot be represented faithfully; the evaluation must be rounded to whichever integer lies nearest, and information about the true, in-between strength of the judgment is lost in the process. This is precisely the shortcoming that the fractional orthopair fuzzy set (FOFS) was designed to remove: by admitting a rational exponent ratio $p/q$ rather than a bare integer, the condition $M^{p/q} + N^{p/q} \leq 1$ allows the admissible region to be tuned continuously rather than in integer steps, and it recovers the IFS, the PFS, and the q-ROFS as special cases of the ratio $p/q$ [12]. Fractional exponents were needed, in short, because integer-only rungs discretise a phenomenon, namely expert confidence, that is inherently continuous, and any model that rounds a fractional judgment to the nearest integer necessarily distorts it.

A second, largely independent line of development concerned not the exponent on $M$ and $N$, but how a single expert evaluation is positioned in the plane at all. Every model discussed so far still commits an expert to one exact pair $(M, N)$, even though real assessments fluctuate within some neighbourhood reflecting residual uncertainty, an imprecision no exponent can capture, since it only reshapes the boundary and says nothing about where the true evaluation lies within it. This is why a circular representation was needed: Atanassov attached a radius to the intuitionistic pair, representing an element by a disk centred on $(M, N)$ rather than the point itself, giving the circular intuitionistic fuzzy set (CIFS) [13]. The same reasoning was carried over to the Pythagorean fuzzy set [14] and, more recently, to the q-ROFS. This last step, the circular $q$-rung orthopair fuzzy set (Cq-ROFS), was needed because the $q$-rung structure alone still fixed each judgment to a single point, and pairing it with a radius $r \in [0, \sqrt{2}]$ let the model absorb the residual imprecision around that point while retaining the wide region already provided by $q$ [15].

These two lines of advancement, one loosening the algebraic exponent and the other enriching the geometric representation, have so far progressed in parallel rather than in combination. Every fractional-order model reviewed above still commits an expert to a single point in the plane, and every circular model reviewed above still restricts its underlying rung parameter to a strictly integer value of $q$. No structure currently in the literature allows a decision-maker to benefit from a continuously tunable fractional exponent and a disk-shaped information region at the same time, even though the reasoning that motivated each extension individually, namely that integer rungs discretise a continuous phenomenon and that a single point cannot capture residual expert imprecision, applies with equal force when both concerns are present together, as they routinely are in practice. Table 1 places the reviewed models on a common footing and makes this gap explicit, motivating the circular fractional orthopair fuzzy set introduced in Section 3.

**Table 1**
Overview of fuzzy set models leading to the proposed circular fractional orthopair fuzzy set

| Model | Constraint | Why the prior model fell short | Contribution of proposed CFOFS |
|---|---|---|---|
| IFS [8] | $M + N \leq 1$ | Plain FS had no channel for non-membership at all | CFOFS retains and refines this dual-grade structure |
| PFS [9] | $M^2 + N^2 \leq 1$ | Linear IFS bound rejected valid pairs with simultaneously high $M, N$ | CFOFS replaces any fixed exponent with a tunable fractional ratio |
| $q$-ROFS [10] | $M^q + N^q \leq 1, q \in \mathbb{Z}^+$ | A single fixed exponent (1 or 2) could not suit every decision context | CFOF generalises the exponent while adding a radius term |
| $pq$-QOFS [11] | $M^p + N^q \leq 1$, $p, q \in \mathbb{Z}^+$ | Shared rung in $q$-ROFS could not scale $M$ and $N$ at different rates | CFOF preserves this decoupling within a fractional ratio |
| FOFS [12] | $M^{p/q} + N^{p/q} \leq 1$, $p, q \in \mathbb{Z}^+, p \geq q$ | Integer-only rungs in earlier models discretised a continuous quantity | CFOFS keeps the fractional ratio but adds a radius for point-level imprecision |
| C$q$-ROFS [15] | $M^q + N^q \leq 1$ with radius $r \in [0, \sqrt{2}]$ | Fractional models still fixed each judgment to a single point | CFOFS combines the same radius with a fractional, rather than integer, exponent |

### 1.2 *Related Studies*

The Schweizer-Sklar (SS) framework of triangular norms and conorms goes back to Schweizer and Sklar's original 1960 formulation [16]. Still, it only attracted sustained attention from the fuzzy decision-making community once researchers realized that its adjustable parameter $\rho$ let it reshape the aggregation curve itself, something ordinary algebraic t-norms could not do. Zhang [17] was among the earliest to carry this into the intuitionistic fuzzy context by using SS aggregation operators (AOs) for IFS, and Wang and Liu [18] further extended it by introducing SS Maclaurin symmetric mean operators that take into account the interrelationships between the criteria. Garg et al. [19] integrated SS aggregation into a complete MADM pipeline using intuitionistic fuzzy data, and Khan et al. [20] used SS aggregation combined with power aggregation and claimed that the combination of SS aggregation and a tunable t-norm can obtain a more balanced group consensus by reducing the extreme expert's opinions. The concept has since spread beyond intuitionistic and Pythagorean environments: Gayen et al. [21] developed a hybrid SS operator for dual hesitant q-rung orthopair fuzzy sets, Wei et al. [22] introduced SS operations and an entropy-weighted Combined Compromise Solution procedure for Fermatean fuzzy green supplier evaluation, and Xue et al. [23] presented SS t-norm operators for picture fuzzy hypersoft sets in a case of green-technology adoption, demonstrating the easy extension of the framework to three or more membership grades.

MULTIMOORA, developed by Brauers and Zavadskas [24], is a method for MADM that incorporates three separate ranking strategies, the ratio system, the reference point approach, and the full multiplicative form, and is less likely to cause rank reversal than single-strategy MADM methods. The method was extended to several uncertain environments: Lin et al. [25] extended it to site selection under picture fuzzy information, Huang et al. [26] extended it to the context of production of solid-state disks with new Dice and Jaccard distance measures, and a further Pythagorean fuzzy variant was applied to novel product development [27]. Earlier, it was extended to interval-valued triangular fuzzy numbers [28] and more recently to a 2-tuple linguistic Pythagorean fuzzy setting [29], all of which have the same motivation: crisp inputs are not always suitable for what a real panel of experts produces. Most extensions are additions to the fuzzy representation, not questioning the ranking logic itself, which is why it has been so enduring in so many contexts, as Hafezalkotob et al. [30] point out in their review.

### *1.3 Research Gap and Motivation*

Despite the continuous development outlined in the previous subsections, there are several methodological shortcomings in how expert uncertainty is flexibly represented in real MADM contexts. The first is that each of the families of circular sets that have been introduced up to now, the CIFS [13], the circular Pythagorean fuzzy set (CPFS) [14], the circular Fermatean fuzzy set (CFFS) [31] and the Cq-ROFS [15] were added to an already integer-constrained structure by attaching a radius to it. The circular wrapper can represent an imprecision *around* a point $(M, N)$, regardless of whether the expert's confidence is at rung $1$ or rung $2$ or $3$ or is left as a general integer $q$; the imprecision is represented by a disk that expands, but the rungs that center it remain unchanged. Second, the FOFS solves this complementary problem by replacing the integer rung with a rational ratio $p/q$ under $M^{p/q} + N^{p/q} \leq 1$ and tuning the admissible boundary continuously so that IFS, PFS, and q-ROFS are recovered as special cases [12]. Yet FOFS, like the $pq$-QOFS before it [11], still commits every judgment to a single exact pair $(M, N)$, with no radius term to absorb the residual uncertainty a circular structure was introduced to capture. Third, no structure in the literature offers both a continuously tunable fractional exponent and a disk-shaped information region, even though both limitations arise jointly whenever a panel evaluates something as multifaceted as a cloud service provider on criteria such as security, scalability, and cost. Fourth, this gap is compounded on the aggregation side: the only operators defined for the Cq-ROFS rest on plain algebraic t-norms with no tunable parameter [15], or on Dombi operators shown to violate monotonicity [32], so even a closed exponent-and-radius gap would leave the aggregation machinery either too rigid or theoretically unsound. Motivated by these observations, the present study introduces the circular fractional orthopair fuzzy set (CFOFS) by replacing the integer-rung q-ROFS underlying the Cq-ROFS with the FOFS, so that $p/q$ tunes the boundary continuously while the retained radius $r$ absorbs residual imprecision; equips it with monotone, parametrically flexible SS operators to close the gap left by [15] and [32]; and embeds the framework in a MULTIMOORA ranking procedure, not previously combined with any circular q-rung orthopair structure, to deliver a decision-support tool for cloud service provider selection that is faithful to fractional expert confidence, sensitive to point-level imprecision, and robust to the rank-reversal weaknesses of single-strategy MADM methods.

### *1.4 Contributions*

The main contributions of this article can be summarised as follows:

1. A new extension of fuzzy set theory, namely the circular fractional orthopair fuzzy set (CFOFS), is introduced by replacing the q-ROFS underlying the Cq-ROFS with the FOFS, and its basic mathematical theory, including its score and accuracy functions and fundamental operational laws, is developed.

2. The theory of the newly proposed set is enriched by formulating SS operations within the CFOFS environment and by developing a corresponding family of SS-based weighted averaging and geometric AOs, along with an investigation of their fundamental properties.

3. The MULTIMOORA method is extended to the CFOFS environment, integrating the ratio system, the reference point approach, and the full multiplicative form within this new setting to produce a more comprehensive and rank-reversal-resistant ranking procedure.

4. The proposed CFOFS-based Schweizer-Sklar MULTIMOORA framework is applied to the problem of cloud computing service provider selection, and its practicality and effectiveness are demonstrated through sensitivity analysis and a comparative study against existing decision-making approaches.

The remaining parts of this article are organized as follows. Section 2 recalls some basic related concepts needed to clearly understand the proposed notion. Section 3 develops the CFOFS and its associated basic theory, and puts forward SS operational laws within this setting. Section 4 introduces a family of AOs based on these SS operational laws and discusses their properties. Section 5 presents a stepwise procedure for the MULTIMOORA method under the CFOFS environment. Section 6 addresses the selection of a suitable cloud computing service provider, along with a parameter analysis. Section 7 carries out a comparative analysis with existing approaches. Finally, Section 8 concludes the article and outlines some directions for future research.

## 2. Theoretical Background

Throughout this paper, $\Upsilon$ stands for a non-empty, finite collection representing the universal set (of alternatives). Furthermore, the symbol $\mathcal{I}$ designates the unit interval $[0,1]$.

The following section outlines the fundamental framework and mathematical definitions pertaining to SS norm, FOFS, and $C_r q - ROF$ sets.

**Definition 1.** *[33] For any pair $\aleph_1, \aleph_2 \in [0,1]$, the Schweizer–Sklar t-norm alongside its corresponding t-conorm are given by:*

$$F_{\xi}(\aleph_1, \aleph_2) = \left(\aleph_1^{\xi} + \aleph_2^{\xi} - 1\right)^{1/\xi}, \tag{1}$$

$$F_{\xi}^{*}(\aleph_1, \aleph_2) = 1 - \left((1-\aleph_1)^{\xi} + (1-\aleph_2)^{\xi} - 1\right)^{1/\xi}, \tag{2}$$

where the parameter satisfies $\xi < 0$. In the limiting case where $\xi = 0$, these operators naturally simplify to $F_{\xi}(\aleph_1, \aleph_2) = \aleph_1\aleph_2$ and $F_{\xi}^{*}(\aleph_1, \aleph_2) = \aleph_1 + \aleph_2 - \aleph_1\aleph_2$.

**Definition 2.** *[12] A FOFS defined over $\Upsilon$ is expressed as*

$$Q = \{(\upsilon, \mathcal{M}(\upsilon), \mathcal{N}(\upsilon)) \,|\, \upsilon \in \Upsilon\}, \tag{3}$$

*in which the mappings $\mathcal{M} : \Upsilon \longrightarrow \mathcal{I}$ and $\mathcal{N} : \Upsilon \longrightarrow \mathcal{I}$ represent the DB and DN, respectively, restricted to $\mathcal{M}^{p/q}(\upsilon) + \mathcal{N}^{p/q}(\upsilon) \leq 1$; $(p, q \in \mathbb{Z}^{+}$such that $p \geq q)$. The relation $\pi(\upsilon) = \left(1 - \left(\mathcal{M}^{p/q}(\upsilon) + \mathcal{N}^{p/q}(\upsilon)\right)\right)^{q/p}$ characterizes the hesitancy degree associated with an element $\upsilon \in \Upsilon$. Additionally, the elements forming a FOFS are referred to as FOFN, which are denoted as $Q = (\mathcal{M}, \mathcal{N})$.*

**Definition 3.** *[12] Consider a family of FOFN given by $Q_{\jmath} = \left(\mathcal{M}_{\jmath}, \mathcal{N}_{\jmath}\right)$ for $\jmath = 1, 2, ..., n$, along with a scalar parameter $\beta > 0$. The basic operational laws are specified by:*

1. $Q_1 \oplus Q_2 = \left(\left(\mathcal{M}_1^{p/q} + \mathcal{M}_2^{p/q} - \mathcal{M}_1^{p/q}\mathcal{M}_2^{p/q}\right)^{q/p}, \mathcal{N}_1\mathcal{N}_2\right);$

2. $Q_1 \otimes Q_2 = \left(\mathcal{M}_1\mathcal{M}_2, \left(\mathcal{N}_1^{p/q} + \mathcal{N}_2^{p/q} - \mathcal{N}_1^{p/q}\mathcal{N}_2^{p/q}\right)^{q/p}\right);$

3. $\beta Q_1 = \left(\left(1-\left(1-\mathcal{M}_1^{p/q}\right)^{\beta}\right)^{q/p}, \mathcal{N}_1^{\beta}\right);$

4. $Q_1^{\beta} = \left(\mathcal{M}_1^{\beta}, \left(1-\left(1-\mathcal{N}_1^{p/q}\right)^{\beta}\right)^{q/p}\right).$

**Definition 4.** *[12] For a given FOFN $Q = (\mathcal{M}, \mathcal{N})$, its score function is calculated according to:*

$$S(Q) = \frac{1 + \mathcal{M}^{p/q} - \mathcal{N}^{p/q}}{2}. \tag{4}$$

**Definition 5.** *[15, 32] A $C_r q - ROFS$ established on $\Upsilon$ constitutes a mathematical structure given by*

$$\mathcal{O} = \{(\upsilon, \mathcal{M}(\upsilon), \mathcal{N}(\upsilon); r(t)) \,|\upsilon \in \Upsilon\} = \{C_r(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon)) \,|\upsilon \in \Upsilon\}, \tag{5}$$

*wherein $\mathcal{M}, \mathcal{N} \in \mathcal{I}$ satisfy the condition $\mathcal{M}^q(\upsilon) + \mathcal{N}^q(\upsilon) \leq 1$, and*
$C_r(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon)) = \left\{(m,n)\,|m,n \in \mathcal{I}\ \&\ \sqrt{(\mathcal{M}(\upsilon)-m)^2 + (\mathcal{N}(\upsilon)-n)^2} \leq r(\upsilon)\right\} \bigcap R;$
$R = \{(m,n) \in \mathcal{I}\ \&\ m^q + n^q \leq 1\}.$

*The quantity $\pi(\upsilon) = (1 - (\mathcal{M}^q(\upsilon) + \mathcal{N}^q(\upsilon)))^{1/q}$ measures the hesitancy degree of an element $\upsilon \in \Upsilon$. Furthermore, individual entities of a $C_r q - ROF$ are designated as $C_r q - ROFN$, which are represented by the notation $\mathcal{O} = (\mathcal{M}, \mathcal{N}; r)$.*

**Definition 6.** *[15] Assuming $\mathcal{O}_l = (\mathcal{M}_l, \mathcal{N}_l; r_l)$ $(l = 1, 2)$ represent two $C_r q - ROFN$, where $\beta > 0$ and $\Cap \in \{\min, \max\}$, the fundamental operations are defined as follows:*

1. $\mathcal{O}_1 \bigcap \mathcal{O}_2 = (\min\{\mathcal{M}_1, \mathcal{M}_2\}, \max\{\mathcal{N}_1, \mathcal{N}_2\}; \Cap\{r_1, r_2\});$
2. $\mathcal{O}_1 \bigcup \mathcal{O}_2 = (\max\{\mathcal{M}_1, \mathcal{M}_2\}, \min\{\mathcal{N}_1, \mathcal{N}_2\}; \Cap\{r_1, r_2\});$
3. $\mathcal{O}_1 \oplus \mathcal{O}_2 = \left(\sqrt[q]{\mathcal{M}_1^q + \mathcal{M}_2^q - \mathcal{M}_1^q\mathcal{M}_2^q}, \mathcal{N}_1\mathcal{N}_2; \Cap\{r_1, r_2\}\right);$
4. $\mathcal{O}_1 \otimes \mathcal{O}_2 = \left(\mathcal{M}_1\mathcal{M}_2, \sqrt[q]{\mathcal{N}_1^q + \mathcal{N}_2^q - \mathcal{N}_1^q\mathcal{N}_2^q}; \Cap\{r_1, r_2\}\right);$
5. $\mathcal{O}_1^c = (\mathcal{N}_1, \mathcal{M}_1; r_1);$

*in which $\mathcal{O}_1^c$ represents the complement associated with $\mathcal{O}_1$.*

**Definition 7.** *[34] For a given $C_r q - ROFN$ denoted by $\mathcal{O} = (\mathcal{M}, \mathcal{N}; r)$, the score function is formulated as*

$$S(\mathcal{O}) = \left(\frac{\mathcal{M}^q - \mathcal{N}^q + r + 1}{2+\sqrt{2}}\right), \tag{6}$$

*in which $S(\mathcal{O}) \in [0, 1]$. Higher values of $S(\mathcal{O})$ correspond to a superior $C_r q - ROFN$ $\mathcal{O}$.*

## 3. Circular Fractional Orthopair Fuzzy Set and Schweizer–Sklar Operations

In this section, we introduce the notion of the proposed fuzzy extension. Along with the definition of the new concept, the related theoretical concepts should also be developed to provide a complete foundation for the proposed extension.

### 3.1 Notion of Circular Fractional Orthopair Fuzzy Set

Definition 2 widens the admissible boundary from an integer rung to a continuously tunable ratio $p/q$, yet it still asks an expert to commit to one exact pair $(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon))$. Definition 5, on the other hand, wraps a disk of radius $r(\upsilon)$ around such a pair, but the disk is only ever centred on a point governed by an integer $q$. The construction introduced below removes both restrictions at once: it centres the disk on a pair governed by the fractional ratio $p/q$, so the model is free to choose, independently of one another, how conservative or permissive its admissible boundary should be and how wide a neighbourhood of imprecision should surround the specific judgment an expert commits to.

**Definition 8.** *A circular fractional orthopair fuzzy set (CFOFS) $\mathscr{L}$ established on $\Upsilon$ constitutes a mathematical structure given by*

$$\mathscr{L} = \{(\upsilon, \mathcal{M}(\upsilon), \mathcal{N}(\upsilon); r(\upsilon)) \,|\upsilon \in \Upsilon\} = \{\mathscr{L}_r(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon)) \,|\upsilon \in \Upsilon\}, \tag{7}$$

*wherein $\mathcal{M}, \mathcal{N} \in \mathcal{I}$ satisfy the fractional admissibility condition*

$$\mathcal{M}^{p/q}(\upsilon) + \mathcal{N}^{p/q}(\upsilon) \leq 1, \qquad p, q \in \mathbb{Z}^+,\ p \geq q, \tag{8}$$

*and*

$$\mathscr{L}_r(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon)) = \left\{(m,n) \,\middle|\, m,n \in \mathcal{I} \,\&\, \sqrt{(\mathcal{M}(\upsilon)-m)^2 + (\mathcal{N}(\upsilon)-n)^2} \leq r(\upsilon)\right\} \bigcap R;$$

$$R = \left\{(m,n) \in \mathcal{I} \,\&\, m^{p/q} + n^{p/q} \leq 1\right\}, \qquad r(\upsilon) \in \left[0, \sqrt{2}\right].$$

*The quantity*

$$\pi(\upsilon) = \left(1 - \left(\mathcal{M}^{p/q}(\upsilon) + \mathcal{N}^{p/q}(\upsilon)\right)\right)^{q/p} \tag{9}$$

*measures the hesitancy degree of an element $\upsilon \in \Upsilon$. Individual entities of a CFOFS are designated as CFOFN, and are denoted by $\mathscr{L} = (\mathcal{M}, \mathcal{N}; r)$.*

**Remark 1.** *Definition 8 does not represent $\upsilon$ by a point but by the disk-shaped region $\mathscr{L}_r(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon))$, so that every pair falling inside this disk is treated as an admissible restatement of the expert's evaluation, while only the centre $(\mathcal{M}(\upsilon), \mathcal{N}(\upsilon))$ is required to satisfy Eq. (8) exactly. Setting $r(\upsilon) = 0$ for every $\upsilon \in \Upsilon$ collapses Definition 8 to the FOFS of Definition 2; setting $p = q$ instead collapses it to the $C_rq$-ROFS of Definition 5 evaluated at $q = 1$. Table 2 records the further special cases obtained under particular choices of $p$, $q$, and $r$.*

**Table 2**
Special cases recovered from the circular fractional orthopair fuzzy set

| Condition on $p, q, r$ | Recovered model | Reference |
|---|---|---|
| $r = 0,\ p = q$ | IFS | [8] |
| $r = 0,\ p = 2q$ | PFS | [9] |
| $r = 0,\ q = 1$ | $q$-ROFS | [10] |
| $r = 0,\ p, q$ arbitrary | FOFS | [12] |
| $r > 0,\ q = 1,\ p = 1$ | CIFS | [13] |
| $r > 0,\ q = 1,\ p = 2$ | Circular PFS | [14] |
| $r > 0,\ p = q$ | $C_rq$-ROFS | [15] |

*A geometric reading of Definition 8*

It is worth being precise about what the fractional exponent contributes here, since it is not extra area. For any $1 \leq p/q \leq 2$, the region $R = \{(m, n) : m^{p/q} + n^{p/q} \leq 1\}$ is nested strictly between the $q = 1$ and $q = 2$ regions, so nothing admissible under a fractional ratio is ever inaccessible to the coarser integer curve $q = 2$; every point reachable at $p/q = 1.5$ was already reachable at $q = 2$. The value of letting $p/q$ vary continuously lies instead in how faithfully a single fixed evaluation is represented once it is judged against the correct curve, and Figure 1 makes this concrete. Take the evaluation $(\mathcal{M}, \mathcal{N}) = (0.60,\ 0.659)$, a pair that sits exactly on the $p/q = 1.5$ boundary. Under $q = 1$ this pair is not admissible at all, since $0.60 + 0.659 = 1.259 > 1$. Judged instead against $p/q = 1.5$, the same pair carries zero hesitancy, $\pi = 0$, because it sits precisely on the boundary the expert's own confidence corresponds to. Forced instead onto $q = 2$, the identical pair is read as carrying hesitancy $\pi \approx 0.454$, as though the expert were still fairly unsure, purely because the curve used to interpret the statement was coarser than the statement itself warranted. Rounding a fractional judgment up to the nearest admissible integer rung therefore does not enlarge what can be expressed; it silently manufactures indecision that was never there.

The same distortion carries over once a radius is attached. Figure 2 fixes one centre $(\mathcal{M}, \mathcal{N}) = (0.60, 0.659)$ and one radius $r = 0.22$, and clips the resulting disk against $R$ under three different rungs. Under $q = 1$ only $4\%$ of the disk survives the clipping in Eq. (7), since the centre itself is barely admissible; under $p/q = 1.5$, the ratio matching the expert's actual confidence, $49\%$ of the disk survives; under $q = 2$, the coarser rung, $78\%$ survives. The disk itself never changed. What changed is how much of the expert's stated imprecision the model is willing to honour once it is forced through the wrong curve. This is precisely the failure mode Definition 8 is built to avoid: by letting $p/q$ track the expert's confidence directly rather than rounding it to the nearest integer rung, the fraction of the disk retained after clipping reflects the imprecision the expert actually intended, neither compressed by too tight a rung nor inflated by too loose one.

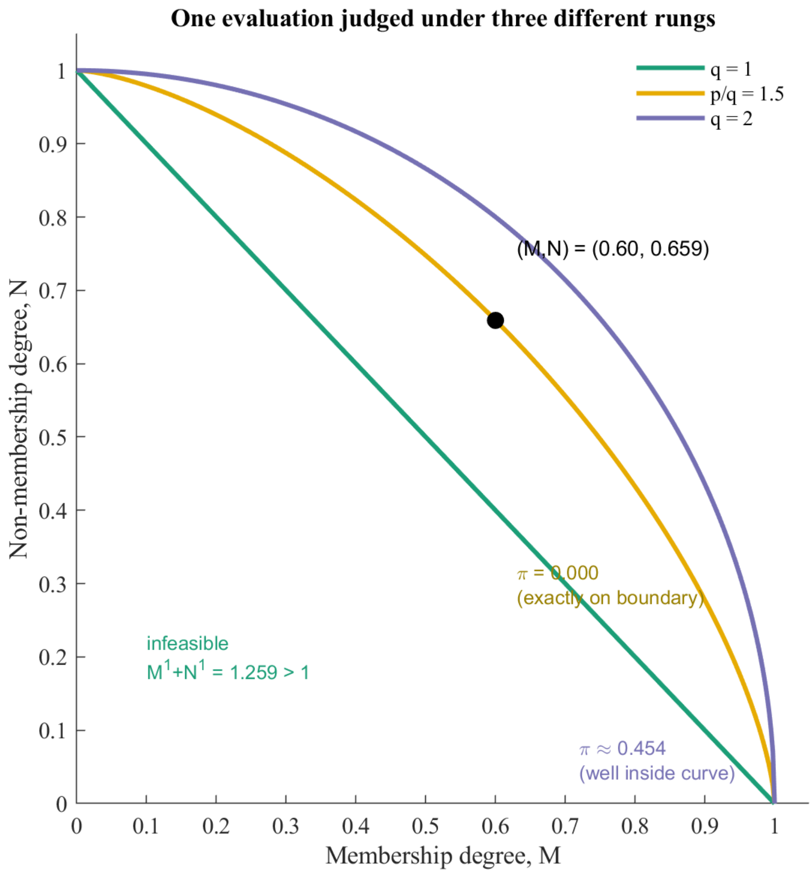


**Fig. 1.** The same evaluation $(\mathcal{M}, \mathcal{N}) = (0.60, 0.659)$ judged against $q = 1$, $p/q = 1.5$, and $q = 2$: it is infeasible under $q = 1$, exactly decisive ($\pi = 0$) under the matching fractional rung, and falsely uncertain ($\pi \approx 0.454$) once rounded up to $q = 2$.

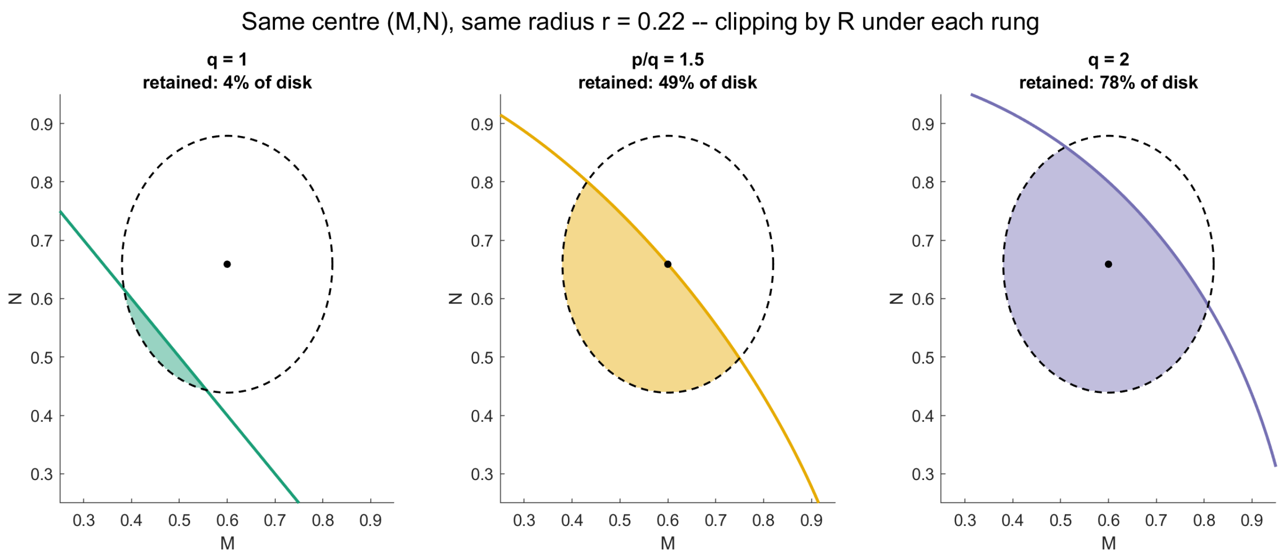


**Fig. 2.** One centre, one radius $r = 0.22$, clipped by $R$ under three rungs: only $4\%$ of the disk survives under $q = 1$, $49\%$ under the matching ratio $p/q = 1.5$, and $78\%$ under $q = 2$ — the same stated imprecision is compressed, honoured, or inflated depending only on which curve it is judged against.

**Definition 9.** *Assume $\mathscr{L}_l = (\mathcal{M}_l, \mathcal{N}_l; r_l)$ $(l = 1, 2)$ represent two CFOFNs governed by a common exponent ratio $p/q$, where $\beta > 0$ and $\Cap \in \{\min, \max\}$. The fundamental operations are defined as follows:*

1. $\mathscr{L}_1 \oplus \mathscr{L}_2 = \left( \left( \mathcal{M}_1^{p/q} + \mathcal{M}_2^{p/q} - \mathcal{M}_1^{p/q} \mathcal{M}_2^{p/q} \right)^{q/p}, \mathcal{N}_1 \mathcal{N}_2; \Cap \{r_1, r_2\} \right);$
2. $\mathscr{L}_1 \otimes \mathscr{L}_2 = \left( \mathcal{M}_1 \mathcal{M}_2, \left( \mathcal{N}_1^{p/q} + \mathcal{N}_2^{p/q} - \mathcal{N}_1^{p/q} \mathcal{N}_2^{p/q} \right)^{q/p}; \Cap \{r_1, r_2\} \right);$
3. $\mathscr{L}_1^c = (\mathcal{N}_1, \mathcal{M}_1; r_1);$
4. $\beta \mathscr{L}_1 = \left( \left( 1 - \left( 1 - \mathcal{M}_1^{p/q} \right)^{\beta} \right)^{q/p}, \mathcal{N}_1^{\beta}; r_1 \right);$
5. $\mathscr{L}_1^{\beta} = \left( \mathcal{M}_1^{\beta}, \left( 1 - \left( 1 - \mathcal{N}_1^{p/q} \right)^{\beta} \right)^{q/p}; r_1 \right).$

*in which $\mathscr{L}_1^c$ denotes the complement of $\mathscr{L}_1$.*

The radius component in items 1–4 is combined through $\Cap \{r_1, r_2\}$ rather than fixed to a single rule, following the same convention adopted for the $C_r q$-ROFS in Definition 6: taking $\Cap = \max$ prevents an aggregation from manufacturing a false sense of precision, since a combined judgment can be no tighter than the vaguer of the two evaluations feeding it, while $\Cap = \min$ remains available where the modeller instead wishes the sharper of two evaluations to dominate.

**Definition 10.** *For a given CFOFN $\mathscr{L} = (\mathcal{M}, \mathcal{N}; r)$, the score function is formulated as*

$$S(\mathscr{L}) = \left( \frac{\mathcal{M}^{p/q} - \mathcal{N}^{p/q} + r + 1}{2 + \sqrt{2}} \right), \tag{10}$$

*in which $S(\mathscr{L}) \in [0, 1]$. Higher values of $S(\mathscr{L})$ correspond to a superior CFOFN $\mathscr{L}$.*

**Definition 11.** *For a given CFOFN $\mathscr{L} = (\mathcal{M}, \mathcal{N}; r)$, the accuracy function is defined as*

$$H(\mathscr{L}) = \mathcal{M}^{p/q} + \mathcal{N}^{p/q} + r. \tag{11}$$

**Definition 12.** *Let $\mathscr{L}_1 = (\mathcal{M}_1, \mathcal{N}_1; r_1)$ and $\mathscr{L}_2 = (\mathcal{M}_2, \mathcal{N}_2; r_2)$ be two CFOFNs. Then:*

1. *if $S(\mathscr{L}_1) > S(\mathscr{L}_2)$, then $\mathscr{L}_1 \succ \mathscr{L}_2$;*
2. *if $S(\mathscr{L}_1) = S(\mathscr{L}_2)$ and $H(\mathscr{L}_1) > H(\mathscr{L}_2)$, then $\mathscr{L}_1 \succ \mathscr{L}_2$;*
3. *if $S(\mathscr{L}_1) = S(\mathscr{L}_2)$ and $H(\mathscr{L}_1) = H(\mathscr{L}_2)$, then $\mathscr{L}_1 = \mathscr{L}_2$.*

*3.2 Schweizer–Sklar Operations*

The following section defines the basic operational laws of CFOFNs in terms of the SS norm and analyzes their key aspects.

**Definition 13.** *Let $\mathscr{L}_\imath = (\mathcal{M}_\imath, \mathcal{N}_\imath; r_\imath)$ $(\imath = 1, 2)$ be two CFOFNs, with $\beta > 0$, $\xi < 0$ and $\Cap \in \{\min, \max\}$. Then the SS operational laws on CFOFNs are outlined as follows:*

1. $\mathscr{L}_1 \oplus \mathscr{L}_2 =$
$$\left(\left(1-\left(\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}+\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-1\right)^{1/\xi}\right)^{q/p},\left(\left(\mathcal{N}_1^{(p/q)\xi}+\mathcal{N}_2^{(p/q)\xi}-1\right)^{1/\xi}\right)^{q/p};\Cap\{r_1,r_2\}\right);$$
2. $\mathscr{L}_1 \otimes \mathscr{L}_2 =$
$$\left(\left(\left(\mathcal{M}_1^{(p/q)\xi}+\mathcal{M}_2^{(p/q)\xi}-1\right)^{1/\xi}\right)^{q/p},\left(1-\left(\left(1-\mathcal{N}_1^{p/q}\right)^{\xi}+\left(1-\mathcal{N}_2^{p/q}\right)^{\xi}-1\right)^{1/\xi}\right)^{q/p};\Cap\{r_1,r_2\}\right);$$
3. $$\beta\mathscr{L}_1 = \left(\left(1-\left(\beta\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p},\left(\left(\beta\mathcal{N}_1^{(p/q)\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p};r_1\right);$$
4. $$\mathscr{L}_1^{\beta} = \left(\left(\left(\beta\mathcal{M}_1^{(p/q)\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p},\left(1-\left(\beta\left(1-\mathcal{N}_1^{p/q}\right)^{\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p};r_1\right).$$

**Theorem 1.** *Let $\mathscr{L}_\imath = (\mathcal{M}_\imath, \mathcal{N}_\imath; r_\imath)$ for $\imath = 1, 2, ..., n$ be collection of CFOFNs, with $\beta, \beta_1 > 0$. Then the following results hold.*

1. $\mathscr{L}_1 \oplus \mathscr{L}_2 = \mathscr{L}_2 \oplus \mathscr{L}_1$;
2. $\mathscr{L}_1 \otimes \mathscr{L}_2 = \mathscr{L}_2 \otimes \mathscr{L}_1$;
3. $\beta\mathscr{L}_1 \oplus \beta\mathscr{L}_2 = \beta(\mathscr{L}_1 \oplus \mathscr{L}_2)$;
4. $(\mathscr{L}_1)^{\beta} \otimes (\mathscr{L}_2)^{\beta} = (\mathscr{L}_1 \otimes \mathscr{L}_2)^{\beta}$;
5. $\beta\mathscr{L}_1 \oplus \beta_1\mathscr{L}_1 = (\beta + \beta_1)\mathscr{L}_1$;
6. $(\mathscr{L}_1)^{\beta} \otimes (\mathscr{L}_1)^{\beta_1} = (\mathscr{L}_1)^{(\beta+\beta_1)}$.

**Proof.** i). $\mathscr{L}_1 \oplus \mathscr{L}_2 =$
$$\left(\left(1-\left(\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}+\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-1\right)^{1/\xi}\right)^{q/p},\left(\left(\mathcal{N}_1^{(p/q)\xi}+\mathcal{N}_2^{(p/q)\xi}-1\right)^{1/\xi}\right)^{q/p};\Cap\{r_1,r_2\}\right) =$$
$$\left(\left(1-\left(\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}+\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-1\right)^{1/\xi}\right)^{q/p},\left(\left(\mathcal{N}_2^{(p/q)\xi}+\mathcal{N}_1^{(p/q)\xi}-1\right)^{1/\xi}\right)^{q/p};\Cap\{r_2,r_1\}\right) =$$
$\mathscr{L}_2 \oplus \mathscr{L}_1$.

Hence, i). holds, and 2) is verified by the similar reasons.

iii). $\beta\mathscr{L}_1 \oplus \beta\mathscr{L}_2 = \left(\left(1-\left(\beta\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_1^{(p/q)\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}; r_1\right) \oplus$
$\left(\left(1-\left(\beta\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_2^{(p/q)\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}; r_2\right) =$

$$\left(\begin{array}{c}\left(1-\left(\beta\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}+\beta\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-(2\beta-1)\right)^{1/\xi}\right)^{q/p}, \\ \left(\left(\beta\mathcal{N}_1^{(p/q)\xi}+\beta\mathcal{N}_2^{(p/q)\xi}-(2\beta-1)\right)^{1/\xi}\right)^{q/p}; \Cap\{r_1, r_2\}\end{array}\right).$$

Now $\beta\left(\mathscr{L}_1 \oplus \mathscr{L}_2\right) =$

$$\beta\left(\left(1-\left(\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}+\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-1\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_1^{(p/q)\xi}+\mathcal{N}_2^{(p/q)\xi}-1\right)^{1/\xi}\right)^{q/p}; \Cap\{r_1, r_2\}\right) =$$

$$\left(\begin{array}{c}\left(1-\left(\beta\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}+\beta\left(1-\mathcal{M}_2^{p/q}\right)^{\xi}-(2\beta-1)\right)^{1/\xi}\right)^{q/p}, \\ \left(\left(\beta\mathcal{N}_1^{(p/q)\xi}+\beta\mathcal{N}_2^{(p/q)\xi}-(2\beta-1)\right)^{1/\xi}\right)^{q/p}; \Cap\{r_1, r_2\}\end{array}\right).$$

Hence, 3). holds, and 4) is verified by the similar reasons.

5). $\beta\mathscr{L}_1 \oplus \beta_1\mathscr{L}_1 = \left(\left(1-\left(\beta\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_1^{(p/q)\xi}-(\beta-1)\right)^{1/\xi}\right)^{q/p}; r_1\right) \oplus$
$\left(\left(1-\left(\beta_1\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta_1-1)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_1^{(p/q)\xi}-(\beta_1-1)\right)^{1/\xi}\right)^{q/p}; r_1\right) =$

$$\left(\begin{array}{c}\left(1-\left((\beta+\beta_1)\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta+\beta_1-1)\right)^{1/\xi}\right)^{q/p}, \left(\left((\beta+\beta_1)\mathcal{N}_1^{(p/q)\xi}-(\beta+\beta_1-1)\right)^{1/\xi}\right)^{q/p} \\ ; \Cap\{r_1, r_2\}\end{array}\right).$$

Now $(\beta+\beta_1)\mathscr{L}_1 =$

$$\left(\begin{array}{c}\left(1-\left((\beta+\beta_1)\left(1-\mathcal{M}_1^{p/q}\right)^{\xi}-(\beta+\beta_1-1)\right)^{1/\xi}\right)^{q/p}, \left(\left((\beta+\beta_1)\mathcal{N}_1^{(p/q)\xi}-(\beta+\beta_1-1)\right)^{1/\xi}\right)^{q/p} \\ ; \Cap\{r_1, r_2\}\end{array}\right).$$

Hence, 5). holds, and 6) is verified by the similar reasons. ■

## 4. Schweizer–Sklar Aggregation Operators

This segment investigates the functional mechanics of the CFOFSSWA and CFOFSSWG operators, adhering to the proposed SS operational laws for CFOFNs.

### *4.1 Schweizer–Sklar Weighted Average Operator*

This part focuses on constructing the CFOFSSWA operator and subsequently examining its key structural properties.

**Definition 14.** *Let $\mathscr{L}_l = \left(\mathcal{M}_l, \mathcal{N}_l; r_l\right)$ ($l = 1, 2, ..., n$) represent an assembly of CFOFNs, accompanied by an associated weight vector $\varpi = (\varpi_1, \varpi_2, ...., \varpi_n)$ satisfying $\varpi_l \in [0, 1]$ and $\sum_{l=1}^{n} \varpi_l = 1$. Under these conditions, the CFOFSSWA operator acts as a mapping $CFOFSSWA : CFOFN^n \longrightarrow CFOFN$ defined by:*

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) = \overset{n}{\underset{l=1}{\oplus}} \varpi_l \mathscr{L}_l = \varpi_1 \mathscr{L}_1 \oplus \varpi_2 \mathscr{L}_2 \oplus ... \oplus \varpi_n \mathscr{L}_n. \tag{12}$$

**Theorem 2.** *Assuming $\mathscr{L}_l = \left(\mathcal{M}_l, \mathcal{N}_l; r_l\right)$ ($l = 1, 2, ..., n$) is a family of CFOFNs and $\xi < 0$, the resulting value obtained through the proposed CFOFSSWA remains a CFOFN, expressed as follows:*

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) = \left(\left(1 - \left(\sum_{l=1}^{n} \varpi_l \left(1 - \mathcal{M}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{l=1}^{n} \varpi_l \mathcal{N}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap \left\{r_1, r_2, ..., r_n\right\}\right). \tag{13}$$

**Proof.** The verification of Eq. (13) proceeds via mathematical induction.

For the base case $n = 2$, we obtain

$$\varpi_1 \mathscr{L}_1 = \left(\left(1 - \left(\varpi_1 \left(1 - \mathcal{M}_1^{p/q}\right)^{\xi} - \left(\varpi_1 - 1\right)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_1^{(p/q)\xi} - \left(\varpi_1 - 1\right)\right)^{1/\xi}\right)^{q/p}; r_1\right),$$

$$\varpi_2 \mathscr{L}_2 = \left(\left(1 - \left(\varpi_2 \left(1 - \mathcal{M}_2^{p/q}\right)^{\xi} - \left(\varpi_2 - 1\right)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_2^{(p/q)\xi} - \left(\varpi_2 - 1\right)\right)^{1/\xi}\right)^{q/p}; r_2\right).$$

Yielding

$$\varpi_1 \mathscr{L}_1 \oplus \varpi_2 \mathscr{L}_2 =$$

$$\left(\begin{array}{c} \left(1 - \left(\varpi_1 \left(1 - \mathcal{M}_1^{p/q}\right)^{\xi} - \left(\varpi_1 - 1\right) + \varpi_2 \left(1 - \mathcal{M}_2^{p/q}\right)^{\xi} - \left(\varpi_2 - 1\right) - 1\right)^{1/\xi}\right)^{q/p}, \\ \left(\left(\varpi_1 \mathcal{N}_1^{(p/q)\xi} - \left(\varpi_1 - 1\right) + \varpi_2 \mathcal{N}_2^{(p/q)\xi} - \left(\varpi_2 - 1\right)\right)^{1/\xi}\right)^{q/p}; \Cap \left\{r_1, r_2\right\} \end{array}\right)$$

$$= \left(\left(1 - \left(\sum_{l=1}^{2} \varpi_l \left(1 - \mathcal{M}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{l=1}^{2} \varpi_l \mathcal{N}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap \left\{r_1, r_2\right\}\right).$$

Consequently, Eq. (13) holds valid for $n = 2$.

Now, assume that Eq. (13) remains valid for $n = k$.

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) =$$

$$\left(\left(1 - \left(\sum_{l=1}^{k} \varpi_l \left(1 - \mathcal{M}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{l=1}^{k} \varpi_l \mathcal{N}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap \left\{r_1, r_2, ..., r_k\right\}\right).$$

To establish the validity of Eq. (13) for $n = k + 1$, we have

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k, \mathscr{L}_{k+1}\right) = CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) \oplus \varpi_{k+1} \mathscr{L}_{k+1} =$$

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) \oplus$$

$$\left(\left(1 - \left(\varpi_{k+1} \left(1 - \mathcal{M}_{k+1}^{p/q}\right)^{\xi} - \left(\varpi_{k+1} - 1\right)\right)^{1/\xi}\right)^{q/p}, \left(\left(\mathcal{N}_{k+1}^{(p/q)\xi} - \left(\varpi_{k+1} - 1\right)\right)^{1/\xi}\right)^{q/p}; r_{k+1}\right)$$

$$= \left(\left(1 - \left(\sum_{l=1}^{k+1} \varpi_l \left(1 - \mathcal{M}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{l=1}^{k+1} \varpi_l \mathcal{N}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap \left\{r_1, r_2, ..., r_{k+1}\right\}\right).$$

This confirms that Eq. (13) holds for $n = k + 1$ as well, thereby completing the verification of Theorem 2. ■

**Theorem 3.** *Consider an collection of CFOFNs denoted by $\mathscr{L}_l = \left(\mathcal{M}_l, \mathcal{N}_l; r_l\right)$ for $l = 1, 2, ..., n$. If every element satisfies $\mathscr{L}_l = \mathscr{L} = \left(\mathcal{M}, \mathcal{N}; r\right)$ for all $l = 1, 2, ..., n$, it follows that*

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) = \mathscr{L}. \tag{14}$$

**Proof.** By applying Eq. (13), it follows that
$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) =$
$$\left(\left(1-\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\left(1-\mathcal{M}_{\mathsf{l}}^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\mathcal{N}_{\mathsf{l}}^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r_1, r_2, ..., r_n\right\}\right)$$
$$= \left(\left(1-\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\left(1-\mathcal{M}^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}, \left(\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\mathcal{N}^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r, r, ..., r\right\}\right) = (\mathcal{M}, \mathcal{N}; r) = \mathscr{L}.$$
This completes the proof of Theorem 3. ∎

**Theorem 4.** *Suppose $\mathscr{L}_{\mathsf{l}} = \left(\mathcal{M}_{\mathsf{l}}, \mathcal{N}_{\mathsf{l}}; r_{\mathsf{l}}\right)$ and $\mathscr{L}_{\mathsf{l}}^* = \left(\mathcal{M}_{\mathsf{l}}^*, \mathcal{N}_{\mathsf{l}}^*; r_{\mathsf{l}}^*\right)$ ($\mathsf{l} = 1, 2, ..., n$) represent two distinct families of CFOFNs. If the inequalities $\mathcal{M}_{\mathsf{l}} \leq \mathcal{M}_{\mathsf{l}}^*$, $\mathcal{N}_{\mathsf{l}} \geq \mathcal{N}_{\mathsf{l}}^*$, and $r_{\mathsf{l}} \leq r_{\mathsf{l}}^*$ hold for all $\mathsf{l} = 1, 2, ..., n$, then*

$$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) \leq CFOFSSWA\left(\mathscr{L}_1^*, \mathscr{L}_2^*, ..., \mathscr{L}_n^*\right). \tag{15}$$

**Proof.** Given that $\mathcal{M}_{\mathsf{l}} \leq \mathcal{M}_{\mathsf{l}}^*$ for each $\mathsf{l}$, it yields $\varpi_{\mathsf{l}}\left(1-\mathcal{M}_{\mathsf{l}}^{p/q}\right)^{\xi} \geq \varpi_{\mathsf{l}}\left(1-\mathcal{M}_{\mathsf{l}}^{*p/q}\right)^{\xi}$
$$\Longrightarrow \left(1-\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\left(1-\mathcal{M}_{\mathsf{l}}^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p} \leq \left(1-\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\left(1-\mathcal{M}_{\mathsf{l}}^{*p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}.$$
Similarly, using the condition $\mathcal{N}_{\mathsf{l}} \geq \mathcal{N}_{\mathsf{l}}^*$, we deduce that
$\left(\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\mathcal{N}_{\mathsf{l}}^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p} \geq \left(\left(\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}}\mathcal{N}_{\mathsf{l}}^{*(p/q)\xi}\right)^{1/\xi}\right)^{q/p}$. Additionally, because $r_{\mathsf{l}} \leq r_{\mathsf{l}}^*$ holds for every $\mathsf{l}$, we have $\Cap\left\{r_1, r_2, ..., r_n\right\} \leq \Cap\left\{r_1^*, r_2^*, ..., r_n^*\right\}$.
Hence, in light of Eq. (10), we conclude that
$CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) \leq CFOFSSWA\left(\mathscr{L}_1^*, \mathscr{L}_2^*, ..., \mathscr{L}_n^*\right)$. ∎

**Theorem 5.** *Consider a sequence of CFOFNs given by $\mathscr{L}_{\mathsf{l}} = \left(\mathcal{M}_{\mathsf{l}}, \mathcal{N}_{\mathsf{l}}; r_{\mathsf{l}}\right)$. Defining the lower and upper boundary elements as $\mathscr{L}_{\min} = \left(\min_{\mathsf{l}}\left\{\mathcal{M}_{\mathsf{l}}\right\}, \max_{\mathsf{l}}\left\{\mathcal{N}_{\mathsf{l}}\right\}; \min_{\mathsf{l}}\left\{r_{\mathsf{l}}\right\}\right)$ and*
*$\mathscr{L}_{\max} = \left(\max_{\mathsf{l}}\left\{\mathcal{M}_{\mathsf{l}}\right\}, \min_{\mathsf{l}}\left\{\mathcal{N}_{\mathsf{l}}\right\}; \max_{\mathsf{l}}\left\{r_{\mathsf{l}}\right\}\right)$ for $\mathsf{l} = 1, 2, ..., n$, respectively, the following inequality holds:*

$$\mathscr{L}_{\min} \leq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) \leq \mathscr{L}_{\max}. \tag{16}$$

**Proof.** Given that $\min_{\mathsf{l}}\left\{\mathcal{M}_{\mathsf{l}}\right\} \leq \mathcal{M}_{\mathsf{l}} \leq \max_{\mathsf{l}}\left\{\mathcal{M}_{\mathsf{l}}\right\}$, $\min_{\mathsf{l}}\left\{\mathcal{N}_{\mathsf{l}}\right\} \leq \mathcal{N}_{\mathsf{l}} \leq \max_{\mathsf{l}}\left\{\mathcal{N}_{\mathsf{l}}\right\}$, and $\min_{\mathsf{l}}\left\{r_{\mathsf{l}}\right\} \leq r_{\mathsf{l}} \leq \max_{\mathsf{l}}\left\{r_{\mathsf{l}}\right\}$ for every $\mathsf{l}$, applying Theorem 4 yields
$CFOFSSWA\left(\mathscr{L}_{\min}, \mathscr{L}_{\min}, ..., \mathscr{L}_{\min}\right) \leq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right)$ alongside
$CFOFSSWA\left(\mathscr{L}_{\max}, \mathscr{L}_{\max}, ..., \mathscr{L}_{\max}\right) \geq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right)$.
Furthermore, utilizing Theorem 3, this reduces to
$\mathscr{L}_{\min} \leq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right)$ and $\mathscr{L}_{\max} \geq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right)$.
Consequently, we arrive at $\mathscr{L}_{\min} \leq CFOFSSWA\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) \leq \mathscr{L}_{\max}$.
This validates Theorem 5. ∎

### *4.2 Schweizer–Sklar Weighted Geometric Operator*

This segment focuses on establishing the CFOFSSWG operator and subsequently evaluating its fundamental characteristics.

**Definition 15.** *Consider an assembly of CFOFNs denoted by $\mathscr{L}_l = \left(\mathcal{M}_l, \mathcal{N}_l; r_l\right)$ ($l = 1, 2, ..., n$), accompanied by a weight vector $\varpi = (\varpi_1, \varpi_2, ...., \varpi_n)$ where $\varpi_l \in [0, 1]$ and the condition $\sum_{l=1}^{n} \varpi_l = 1$ holds. The CFOFSSWG operator then serves as a mapping $CFOFSSWG : CFOFN^n \longrightarrow CFOFN$ formulated as:*

$$CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) = \underset{l=1}{\overset{n}{\otimes}} \mathscr{L}_l^{\varpi_l} = \mathscr{L}_1^{\varpi_1} \otimes \mathscr{L}_2^{\varpi_2} \otimes ... \otimes \mathscr{L}_n^{\varpi_n}. \tag{17}$$

**Theorem 6.** *Let $\mathscr{L}_l = \left(\mathcal{M}_l, \mathcal{N}_l; r_l\right)$ ($l = 1, 2, ..., n$) be a set of CFOFNs, and let $\xi < 0$. Then, the value aggregated via the proposed operator likewise forms a CFOFN, structured as follows:*

$$CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n\right) = \left(\left(\left(\sum_{l=1}^{n} \varpi_l \mathcal{M}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\sum_{l=1}^{n} \varpi_l \left(1 - \mathcal{N}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r_1, r_2, ..., r_n\right\}\right). \tag{18}$$

**Proof.** We establish Eq. (18) through the principle of mathematical induction.

For the base step where $n = 2$, we have

$$\mathscr{L}_1^{\varpi_1} = \left(\left(\left(\mathcal{M}_1^{(p/q)\xi} - (\varpi_1 - 1)\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\varpi_1\left(1 - \mathcal{N}_1^{p/q}\right)^{\xi} - (\varpi_1 - 1)\right)^{1/\xi}\right)^{q/p}; r_1\right),$$

$$\mathscr{L}_2^{\varpi_2} = \left(\left(\left(\mathcal{M}_2^{(p/q)\xi} - (\varpi_2 - 1)\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\varpi_2\left(1 - \mathcal{N}_2^{p/q}\right)^{\xi} - (\varpi_2 - 1)\right)^{1/\xi}\right)^{q/p}; r_2\right).$$

This yields

$$\mathscr{L}_1^{\varpi_1} \otimes \mathscr{L}_2^{\varpi_2} = \begin{pmatrix} \left(\left(\varpi_1 \mathcal{M}_1^{(p/q)\xi} - (\varpi_1 - 1) + \varpi_2 \mathcal{M}_2^{(p/q)\xi} - (\varpi_2 - 1)\right)^{1/\xi}\right)^{q/p}, \\ \left(1 - \left(\varpi_1\left(1 - \mathcal{N}_1^{p/q}\right)^{\xi} - (\varpi_1 - 1) + \varpi_2\left(1 - \mathcal{N}_2^{p/q}\right)^{\xi} - (\varpi_2 - 1) - 1\right)^{1/\xi}\right)^{q/p} \\ ; \Cap\left\{r_1, r_2\right\} \end{pmatrix}$$

$$= \left(\left(\left(\sum_{l=1}^{2} \varpi_l \mathcal{M}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\sum_{l=1}^{2} \varpi_l \left(1 - \mathcal{N}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r_1, r_2\right\}\right).$$

Hence, Eq. (18) holds for $n = 2$.

Assume that Eq. (18) remains valid for $n = k$:

$$CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) = \left(\left(\left(\sum_{l=1}^{k} \varpi_l \mathcal{M}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\sum_{l=1}^{k} \varpi_l \left(1 - \mathcal{N}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r_1, r_2, ..., r_k\right\}\right).$$

To verify Eq. (18) for $n = k + 1$, we examine

$$CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k, \mathscr{L}_{k+1}\right) = CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) \otimes \mathscr{L}_{k+1}^{\varpi_{k+1}} =$$

$$CFOFSSWG\left(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_k\right) \otimes$$

$$\left(\left(\left(\mathcal{M}_{k+1}^{(p/q)\xi} - (\varpi_{k+1} - 1)\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\varpi_{k+1}\left(1 - \mathcal{N}_{k+1}^{p/q}\right)^{\xi} - (\varpi_{k+1} - 1)\right)^{1/\xi}\right)^{q/p}; r_{k+1}\right)$$

$$= \left(\left(\left(\sum_{l=1}^{k+1} \varpi_l \mathcal{M}_l^{(p/q)\xi}\right)^{1/\xi}\right)^{q/p}, \left(1 - \left(\sum_{l=1}^{k+1} \varpi_l \left(1 - \mathcal{N}_l^{p/q}\right)^{\xi}\right)^{1/\xi}\right)^{q/p}; \Cap\left\{r_1, r_2, ..., r_{k+1}\right\}\right).$$

This confirms that Eq. (18) is satisfied for $n = k + 1$. Accordingly, Theorem 6 is proven. ■

**Theorem 7.** *Given a set of CFOFNs denoted by $\mathscr{L}_\text{l} = (\mathcal{M}_\text{l}, \mathcal{N}_\text{l}; r_\text{l})$ for $\text{l} = 1, 2, ..., n$, if each element satisfies $\mathscr{L}_\text{l} = \mathscr{L} = (\mathcal{M}, \mathcal{N}; r)$ across all $\text{l} = 1, 2, ..., n$, then*

$$CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) = \mathscr{L}. \tag{19}$$

**Proof.** By applying Eq. (18), we derive

$$CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) = \left( \left( \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \mathcal{M}_\text{l}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p}, \left( 1 - \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \left( 1 - \mathcal{N}_\text{l}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p}; \Cap \{r_1, r_2, ..., r_n\} \right)$$

$$= \left( \left( \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \mathcal{M}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p}, \left( 1 - \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \left( 1 - \mathcal{N}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p}; \Cap \{r, r, ..., r\} \right) = (\mathcal{M}, \mathcal{N}; r) = \mathscr{L}.$$

This completes the verification of Theorem 7. ■

**Theorem 8.** *Let $\mathscr{L}_\text{l} = (\mathcal{M}_\text{l}, \mathcal{N}_\text{l}; r_\text{l})$ and $\mathscr{L}_\text{l}^* = (\mathcal{M}_\text{l}^*, \mathcal{N}_\text{l}^*; r_\text{l}^*)$ ($\text{l} = 1, 2, ..., n$) be two distinct families of CFOFNs. If $\mathcal{M}_\text{l} \leq \mathcal{M}_\text{l}^*$, $\mathcal{N}_\text{l} \geq \mathcal{N}_\text{l}^*$, and $r_\text{l} \leq r_\text{l}^*$ hold for each $\text{l} = 1, 2, ..., n$, then*

$$CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) \leq CFOFSSWG(\mathscr{L}_1^*, \mathscr{L}_2^*, ..., \mathscr{L}_n^*). \tag{20}$$

**Proof.** Owing to the condition $\mathcal{M}_\text{l} \leq \mathcal{M}_\text{l}^*$, it follows that

$$\left( \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \mathcal{M}_\text{l}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p} \leq \left( \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} {\mathcal{M}^*}_\text{l}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p}.$$

In addition, given that $\mathcal{N}_\text{l} \geq \mathcal{N}_\text{l}^*$ for every $\text{l}$, we obtain $\varpi_\text{l} \left( 1 - \mathcal{N}_\text{l}^{p/q} \right)^{\xi} \leq \varpi_\text{l} \left( 1 - {\mathcal{N}^*}_\text{l}^{p/q} \right)^{\xi}$

$$\Longrightarrow \left( 1 - \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \left( 1 - \mathcal{N}_\text{l}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p} \geq \left( 1 - \left( \sum_{\text{l}=1}^{n} \varpi_\text{l} \left( 1 - {\mathcal{N}^*}_\text{l}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p}.$$

Furthermore, since $r_\text{l} \leq r_\text{l}^*$ holds across all $\text{l}$, we have $\Cap \{r_1, r_2, ..., r_n\} \leq \Cap \{r_1^*, r_2^*, ..., r_n^*\}$.
By combining these inequalities in accordance with Eq. (10), it yields
$CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) \leq CFOFSSWG(\mathscr{L}_1^*, \mathscr{L}_2^*, ..., \mathscr{L}_n^*)$. ■

**Theorem 9.** *Consider a collection of CFOFNs given by $\mathscr{L}_\text{l} = (\mathcal{M}_\text{l}, \mathcal{N}_\text{l}; r_\text{l})$. Defining the lower and upper bounding elements as $\mathscr{L}_{\min} = (\min_\text{l} \{\mathcal{M}_\text{l}\}, \max_\text{l} \{\mathcal{N}_\text{l}\}; \min_\text{l} \{r_\text{l}\})$ and $\mathscr{L}_{\max} = (\max_\text{l} \{\mathcal{M}_\text{l}\}, \min_\text{l} \{\mathcal{N}_\text{l}\}; \max_\text{l} \{r_\text{l}\})$ for $\text{l} = 1, 2, ..., n$, respectively, the following inequality is satisfied:*

$$\mathscr{L}_{\min} \leq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) \leq \mathscr{L}_{\max}. \tag{21}$$

**Proof.** Given that $\min_\text{l} \{\mathcal{M}_\text{l}\} \leq \mathcal{M}_\text{l} \leq \max_\text{l} \{\mathcal{M}_\text{l}\}$, $\min_\text{l} \{\mathcal{N}_\text{l}\} \leq \mathcal{N}_\text{l} \leq \max_\text{l} \{\mathcal{N}_\text{l}\}$, and $\min_\text{l} \{r_\text{l}\} \leq r_\text{l} \leq \max_\text{l} \{r_\text{l}\}$ hold for every $\text{l}$, applying Theorem 8 establishes
$CFOFSSWG(\mathscr{L}_{\min}, \mathscr{L}_{\min}, ..., \mathscr{L}_{\min}) \leq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n)$ alongside
$CFOFSSWG(\mathscr{L}_{\max}, \mathscr{L}_{\max}, ..., \mathscr{L}_{\max}) \geq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n)$.
Furthermore, by virtue of Theorem 7, this simplifies to
$\mathscr{L}_{\min} \leq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n)$ and $\mathscr{L}_{\max} \geq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n)$.
Consequently, we obtain $\mathscr{L}_{\min} \leq CFOFSSWG(\mathscr{L}_1, \mathscr{L}_2, ..., \mathscr{L}_n) \leq \mathscr{L}_{\max}$.
This verifies Theorem 9. ■

## 5. Proposed Decision Framework

This section investigates the practical utility of the CFOFSSWA and CFOFSSWG operators within a MADM context. To evaluate the robustness of the formulated theoretical model, the MULTIMOORA technique is blended with these novel aggregation operators. For this purpose, consider a finite set of potential alternatives denoted by $\mathsf{Z} = \{\mathsf{Z}_i; i = 1, 2, ..., m\}$ together with an associated family of evaluation attributes $\mathbb{A} = \{\mathbb{A}_{\mathsf{l}}; \mathsf{l} = 1, 2, ..., n\}$. The performance assessment of alternative $\mathsf{Z}_i$ across criterion $\mathbb{A}_{\mathsf{l}}$ is captured as a CFOFN $\mathscr{L}_{i\mathsf{l}} = (\mathcal{M}_{i\mathsf{l}}, \mathcal{N}_{i\mathsf{l}}; r_{i\mathsf{l}})$, forming the initial decision matrix $M = [\mathscr{L}_{i\mathsf{l}}]_{m\times n}$. The criterion weighting scheme is represented by the column vector $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$, which obeys the standard constraints $\varpi_{\mathsf{l}} \in [0, 1]$ and $\sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}} = 1$. The sequential steps of the extended MULTIMOORA methodology tailored for CFOF environments are elaborated below.

*Step 1: Normalization:* Attributes within MADM models typically fall into two categories: benefit-oriented and cost-oriented. To eliminate dimensional disparities across heterogeneous attributes, all values must be standardized into a uniform format. Accordingly, the decision matrix entries $\mathscr{L}_{i\mathsf{l}}$ assigned by domain experts undergo normalization via Eq. (22).

$$\breve{\mathscr{L}}_{i\mathsf{l}} = \begin{cases} (\mathcal{M}_{i\mathsf{l}}, \mathcal{N}_{i\mathsf{l}}; r_{i\mathsf{l}}), & \text{if } \mathbb{A}_{\mathsf{l}} \text{ is benefit type} \\ (\mathcal{N}_{i\mathsf{l}}, \mathcal{M}_{i\mathsf{l}}; r_{i\mathsf{l}}), & \text{if } \mathbb{A}_{\mathsf{l}} \text{ is cost type.} \end{cases} \tag{22}$$

*Step 2: CFOF ratio system:* *Step 2(i):* Synthesize the CFOFN evaluations of each alternative into consolidated values by applying the formulated CFOFSSWA operator, expressed as:

$$\mathscr{L}_i = CFOFSSWA(\mathscr{L}_{i\mathsf{l}}, \mathscr{L}_{i\mathsf{l}}, ..., \mathscr{L}_{in}) = \left( \left( 1 - \left( \sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}} \left( 1 - \mathcal{M}_{i\mathsf{l}}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p}, \left( \left( \sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}} \mathcal{N}_{i\mathsf{l}}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p}; \Cap \{r_1, r_2, ..., r_n\} \right). \tag{23}$$

*Step 2(ii):* Compute the score function value $\overline{S}(\mathscr{L}_i)$ for every candidate alternative using Eq. (10).

*Step 2(iii):* Establish the preferential ordering of alternatives, where a superior score $\overline{S}(\mathscr{L}_i)$ corresponds to a more favorable alternative $\mathsf{Z}_i$, specified by

$$\mathsf{Z}_{RS} = \left\{ \mathsf{Z}_i | \max_i \overline{S}(\mathscr{L}_i) \right\}. \tag{24}$$

*Step 3: CFOF reference point:* The essential objective of this stage is to establish a suitable benchmark point and subsequently quantify the deviation of each candidate option from this reference. The procedure unfolds across three consecutive sub-steps:

*Step 3(i):* Construct the CFOF reference point $P = (P_1, P_2, ..., P_n)$ utilizing Eq. (25)

$$P_{\mathsf{l}} = (\mathcal{M}_{P\mathsf{l}}, \mathcal{N}_{P\mathsf{l}}, r_{P\mathsf{l}}) = \left( \max_i \mathcal{M}_{i\mathsf{l}}, \min_i \mathcal{N}_{i\mathsf{l}}; \max_i r_{i\mathsf{l}} \right). \tag{25}$$

*Step 3(ii):* Evaluate the spatial distance connecting the benchmark points to the respective attributes of each alternative via Eq. (26)

$$\overline{\overline{S}}(\mathscr{L}_i) = d(\mathscr{L}_{i\mathsf{l}}, P_{\mathsf{l}}) = \frac{1}{2n} \sum_{\mathsf{l}=1}^{n} \left( \frac{|r_{i\mathsf{l}} - r_{P\mathsf{l}}|}{\sqrt{2}} + \frac{1}{2} \left( \left| \mathcal{M}_{i\mathsf{l}}^{p/q} - \mathcal{M}_{P\mathsf{l}}^{p/q} \right| + \left| \mathcal{N}_{i\mathsf{l}}^{p/q} - \mathcal{N}_{P\mathsf{l}}^{p/q} \right| \right) \right). \tag{26}$$

*Step 3(iii):* Order the alternatives under the principle that a smaller distance value $\overline{\overline{S}}(\mathscr{L}_i)$ signifies a more optimal choice $\mathsf{Z}_i$, namely

$$\mathsf{Z}_{RP} = \left\{ \mathsf{Z}_i | \min_i \overline{\overline{S}}(\mathscr{L}_i) \right\}. \tag{27}$$

*Step 4: CFOF full multiplicative form:* *Step 4(i):* Synthesize the CFOFN evaluations of each candidate option into single aggregated values using the proposed CFOFSSWG operator, expressed as follows:

$$\mathscr{L}_i = CFOFSSWG\left(\mathscr{L}_{i\mathsf{l}}, \mathscr{L}_{i\mathsf{l}}, ..., \mathscr{L}_{in}\right) =$$
$$\left( \left( \left( \left( \sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}} \mathcal{M}_{i\mathsf{l}}^{(p/q)\xi} \right)^{1/\xi} \right)^{q/p}, \left( 1 - \left( \sum_{\mathsf{l}=1}^{n} \varpi_{\mathsf{l}} \left( 1 - \mathcal{N}_{i\mathsf{l}}^{p/q} \right)^{\xi} \right)^{1/\xi} \right)^{q/p} ; \Cap \{r_1, r_2, ..., r_n\} \right) \right). \tag{28}$$

*Step 4(ii):* Determine the score value $\overline{\overline{\overline{S}}}(\mathscr{L}_i)$ for every alternative in light of Eq. (10).

*Step 4(iii):* Order the alternatives under the criterion that an elevated score $\overline{\overline{\overline{S}}}(\mathscr{L}_i)$ signifies a superior candidate $\mathsf{Z}_i$, specified by

$$\mathsf{Z}_{FM} = \left\{ \mathsf{Z}_i | \max_i \overline{\overline{\overline{S}}}(\mathscr{L}_i) \right\}. \tag{29}$$

*Step 5: Borda rule:* To reconcile the individual orderings $\mathsf{Z}_{RS}$, $\mathsf{Z}_{RP}$, and $\mathsf{Z}_{FM}$ obtained across the three distinct subsystems for each alternative $\mathsf{Z}_i$ $(i = 1, 2, ..., m)$, an enhanced Borda count method [35] is employed to construct a unified preference structure. The computation scheme proceeds as follows:

$$\overline{S}\left(\mathscr{L}_i'\right) = \frac{\overline{S}(\mathscr{L}_i)}{\sqrt{\sum_{i=1}^{m} \left(\overline{S}(\mathscr{L}_i)\right)^2}}, \tag{30}$$

$$\overline{\overline{S}}\left(\mathscr{L}_i'\right) = \frac{\overline{\overline{S}}(\mathscr{L}_i)}{\sqrt{\sum_{i=1}^{m} \left(\overline{\overline{S}}(\mathscr{L}_i)\right)^2}}, \tag{31}$$

$$\overline{\overline{\overline{S}}}\left(\mathscr{L}_i'\right) = \frac{\overline{\overline{\overline{S}}}(\mathscr{L}_i)}{\sqrt{\sum_{i=1}^{m} \left(\overline{\overline{\overline{S}}}(\mathscr{L}_i)\right)^2}}. \tag{32}$$

The definitive preference value for each alternative is established through the formulation below:

$$\Xi_i = \overline{S}\left(\mathscr{L}_i'\right) \frac{m - \overline{O}(\mathsf{Z}_i) + 1}{m(m+1)/2} - \overline{\overline{S}}\left(\mathscr{L}_i'\right) \frac{\overline{\overline{O}}(\mathsf{Z}_i)}{m(m+1)/2} + \overline{\overline{\overline{S}}}\left(\mathscr{L}_i'\right) \frac{m - \overline{\overline{\overline{O}}}(\mathsf{Z}_i) + 1}{m(m+1)/2}, \tag{33}$$

where $\overline{O}(\mathsf{Z}_i)$, $\overline{\overline{O}}(\mathsf{Z}_i)$, and $\overline{\overline{\overline{O}}}(\mathsf{Z}_i)$ denote the ordinal positions of alternative $\mathsf{Z}_i$ derived from the ratio system, reference point, and full multiplicative models, respectively. Candidates achieving higher overall index values reflect optimal choices, dictating that the final hierarchy of alternatives be arranged in non-increasing order of their respective performance values.

## 6. Case Study: Cloud Service Provider Selection

To illustrate the utility of the proposed framework, this section presents a numerical example on selecting a suitable cloud service provider. A CFOFN approach is employed for evaluating the alternatives.

### *6.1 Problem Description*

For cloud migrations of mission-critical applications, price would not be the sole determining factor in choosing the vendor. While a cheap vendor might fail to provide adequate data security, which could lead to higher regulatory costs that outweigh the savings in hosting charges, a vendor that provides secure cloud services but lacks scalability might prevent the firm from managing increased traffic volume due to product launches or peak seasons. Thus, the selection of cloud service vendors constitutes a multi-criteria decision-making problem and becomes more complex due to divergent opinions among IT managers, purchasing managers, and possibly outside consultants regarding the different vendors. The evaluations depend on historical outage experience, the credibility of vendors' certification compliance, and variable budgets across fiscal periods, among others.

This case study concerns a company's migration to an external cloud platform to transfer its existing data infrastructure and applications to the cloud. After the technical screening of several vendors, six remain meeting the set criteria. The selection process will involve ranking the vendors based on the proposed CFOFS-based SS MULTIMOORA approach.

Suppose the set of alternatives representing the candidate cloud service providers is given by $Z = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$ with each alternative being briefly described below:

1. $Z_1$ **- NimbusCore Cloud:** A proven enterprise-grade player, with a strong focus on compliance depth and platform resilience. It has a good security certification portfolio, has maintained its published uptime guarantees close to what it has delivered over the last few years, and has a limited regional data-center presence compared to some competitors, with pricing slightly above the market median.

2. $Z_2$ **- Vertex CloudWorks:** A performance-focused vendor focused on auto-scaling and a large variety of compute tiers, which is appealing to clients who have volatile or seasonal traffic. It is competitively priced and has a clear cost structure, but its security tooling and audit trail are less robust than those of some of the more compliance-oriented vendors on the list.

3. $Z_3$ **- Stratus Nine Systems:** Mid-market, yet with a focus on cost discipline but not skimping on the fundamentals that make any service trustworthy: strong encryption, decent support responsiveness, and one of the more predictable billing models among the candidates. It's not elastic, with its scaling tiers being coarser and slower to provision than the performance-first vendors.

4. $Z_4$ **- OrbitalGrid Cloud:** A newer player, but growing its compliance track record and support system, for which the panel's uncertainty about security is a reflection. It makes up for this with good scalability and competitive introductory pricing, designed to capture market share from the more mature vendors.

5. $Z_5$ **- Ironhost Cloud Services:** A provider that is more oriented towards dedicated and semi-dedicated infrastructure instead of fully shared multi-tenant environments, and therefore enjoys a strong reputation for security isolation and reliable uptime. The compromise is between elasticity and cost: adding more capacity takes longer than with cloud-native competitors, and costs are higher.

6. $Z_6$ **- Meridian Cloud Technologies:** A generalist vendor with competitive pricing, simple terms, and pretty good performance, but a reputation for support and incident response that hasn't been consistent enough for the panel to feel confident in its service level obligations.

The selection of the most suitable provider depends on four critical factors (attributes), defined as:

1. $\mathbb{A}_1$ **- Security and Data Privacy:** This criterion reflects the ability of the provider to ensure the security of tenant data and its protection against unauthorized access. It addresses encryption standards at rest and in transit, identity and access management controls, third-party security certifications (such as ISO 27001 or SOC 2), and the provider's history of breach disclosures. A vendor with a solid history of compliance and robust technical security measures is favorably rated. In contrast, one that has experienced incidents or whose audits are not transparent is penalized for its compliance issues, despite its other competitive products.

2. $\mathbb{A}_2$ **- Scalability and Performance:** This criterion indicates a provider's capacity to handle sudden and sustained increases in workload without a commensurate decrease in responsiveness. It covers the wide range of compute and storage tiers, How quickly it can scale up and down, the geographic distance of data centers from the client's user base, and measured latency and throughput under load. A provider that can scale elastically and maintain low latency across regions is rated higher than one with fixed capacity, or that experiences performance drops as demand increases.

3. $\mathbb{A}_3$ **- Cost-effectiveness:** This criterion considers the cost of the provider and the value that is provided. Subscription and pay-as-you-go pricing, hidden fees like data egress and API call fees, discount models for long-term agreements, and the visibility of the pricing model. Providers should have a predictable pricing structure and avoid hidden costs that make the total cost of ownership too high as usage increases.

4. $\mathbb{A}_4$ **- Service Reliability and Support:** This criterion measures the reliability of a provider's services and the provider's ability to respond when something goes wrong. It relies on published uptime guarantees and penalties for failures, past downtime data, responsiveness of technical support channels, and the transparency of incident post-mortems. A provider with robust service level agreements that are adhered to is better than one whose service level agreements are not met.

The relative importance of these attributes, expressed as a weight vector, is given by: $\varpi = (0.30, 0.25, 0.20, 0.25)^T$, reflecting the shortlisting committee's judgment that security and reliability carry somewhat greater weight in this migration than raw cost or scalability, without either of the latter two being treated as negligible.

To evaluate the alternatives against the attributes, expert opinions were solicited from the procurement panel, and their consolidated assessments, expressed as CFOFNs, are detailed in matrix $M$.

$$M = \begin{array}{c} \\ \mathsf{Z}_1 \\ \mathsf{Z}_2 \\ \mathsf{Z}_3 \\ \mathsf{Z}_4 \\ \mathsf{Z}_5 \\ \mathsf{Z}_6 \end{array} \left( \begin{array}{cccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 \\ (0.80, 0.30; 0.15) & (0.55, 0.40; 0.02) & (0.50, 0.50; 0.12) & (0.68, 0.30; 0.18) \\ (0.60, 0.50; 0.02) & (0.70, 0.38; 0.10) & (0.75, 0.40; 0.15) & (0.55, 0.40; 0.20) \\ (0.60, 0.20; 0.08) & (0.50, 0.50; 0.20) & (0.70, 0.22; 0.08) & (0.55, 0.25; 0.04) \\ (0.40, 0.50; 0.10) & (0.60, 0.35; 0.15) & (0.65, 0.45; 0.20) & (0.48, 0.30; 0.02) \\ (0.65, 0.35; 0.05) & (0.55, 0.22; 0.04) & (0.65, 0.40; 0.08) & (0.68, 0.25; 0.10) \\ (0.70, 0.40; 0.06) & (0.65, 0.36; 0.08) & (0.70, 0.42; 0.15) & (0.60, 0.60; 0.02) \end{array} \right)$$

### *6.2 Evaluation Procedure*

In the sequence below, the step-by-step execution of the presented MULTIMOORA algorithm (configured with parameter choices $\xi = -3$, $p = 3$, $q = 2$, and $\Cap = \max$) is applied to the input information contained in matrix $M$.

*Step 1:* The normalization stage is bypassed because all evaluation attributes share a uniform classification.

*Step 2(i):* Utilizing Eq. (23), the aggregated evaluations are computed as: $\mathscr{L}_1 = (0.7255, 0.3314; 0.1800)$, $\mathscr{L}_2 = (0.6751, 0.4121; 0.2000)$, $\mathscr{L}_3 = (0.6085, 0.2307; 0.2000)$, $\mathscr{L}_4 = (0.5581, 0.3605; 0.2000)$, $\mathscr{L}_5 = (0.6429, 0.2639; 0.1000)$, $\mathscr{L}_6 = (0.6713, 0.4092; 0.1500)$.

*Step 2(ii):* Through Eq. (10), the score values for each $\mathscr{L}_i$ are determined as: $\overline{S}(\mathscr{L}_1) = 0.4707$, $\overline{S}(\mathscr{L}_2) = 0.4364$, $\overline{S}(\mathscr{L}_3) = 0.4580$, $\overline{S}(\mathscr{L}_4) = 0.4102$, $\overline{S}(\mathscr{L}_5) = 0.4335$, $\overline{S}(\mathscr{L}_6) = 0.4213$.

*Step 2(iii):* Based on the computed score values, the candidate ordering is established as: $\mathsf{Z}_1 > \mathsf{Z}_3 > \mathsf{Z}_2 > \mathsf{Z}_5 > \mathsf{Z}_6 > \mathsf{Z}_4$.

*Step 3(i):* Applying Eq. (25) yields the benchmark reference profile:

$P = ((0.8000, 0.2000; 0.1500), (0.7000, 0.2200; 0.2000), 0.7500, 0.2200; 0.2000), (0.6800, 0.2500; 0.2000))$.

*Step 3(ii):* The separation distances linking the reference point to every individual candidate are evaluated using Eq. (26), giving:

$\overline{\overline{S}}(\mathscr{L}_1) = 0.0865$, $\overline{\overline{S}}(\mathscr{L}_2) = 0.0920$, $\overline{\overline{S}}(\mathscr{L}_3) = 0.0903$, $\overline{\overline{S}}(\mathscr{L}_4) = 0.1212$, $\overline{\overline{S}}(\mathscr{L}_5) = 0.0901$, $\overline{\overline{S}}(\mathscr{L}_6) = 0.1099$,

*Step 3(iii):* Ordering the values of $\overline{\overline{S}}(\mathscr{L}_i)$ in ascending sequence produces: $Z_1 > Z_5 > Z_3 > Z_2 > Z_6 > Z_4$.

*Step 4(i):* Applying Eq. (28), the consolidated values are calculated as: $\mathscr{L}_1 = (0.5937, 0.3867; 0.1800)$, $\mathscr{L}_2 = (0.6192, 0.4340; 0.2000)$, $\mathscr{L}_3 = (0.5624, 0.3451; 0.2000)$, $\mathscr{L}_4 = (0.4724, 0.4233; 0.2000)$, $\mathscr{L}_5 = (0.6203, 0.3166; 0.1000)$, $\mathscr{L}_6 = (0.6549, 0.4815; 0.1500)$.

*Step 4(ii):* Via Eq. (10), the corresponding score outputs for $\mathscr{L}_i$ are given by: $\overline{\overline{\overline{S}}}(\mathscr{L}_1) = 0.4092$, $\overline{\overline{\overline{S}}}(\mathscr{L}_2) = 0.4104$, $\overline{\overline{\overline{S}}}(\mathscr{L}_3) = 0.4156$, $\overline{\overline{\overline{S}}}(\mathscr{L}_4) = 0.3659$, $\overline{\overline{\overline{S}}}(\mathscr{L}_5) = 0.4131$, $\overline{\overline{\overline{S}}}(\mathscr{L}_6) = 0.3942$.

*Step 4(iii):* Considering the above score assessments, the alternative hierarchy is ordered as: $Z_3 > Z_5 > Z_2 > Z_1 > Z_6 > Z_4$.

*Step 5:* Executing Eqs. (30)–(33) produces the final utility values $\Xi_1 = 0.1675$, $\Xi_2 = 0.0846$, $\Xi_3 = 0.1690$, $\Xi_4 = -0.1067$, $\Xi_5 = 0.1222$ and $\Xi_6 = -0.0323$. Consequently, the consolidated ranking sequence of alternatives is $Z_3 > Z_1 > Z_5 > Z_2 > Z_6 > Z_4$.

### 6.3 Sensitivity Analysis

This section conducts a sensitivity analysis to evaluate how variations in the system parameters $\xi$ and $q$ influence the stability and performance of the model outcomes.

#### 6.3.1 Sensitivity regarding $\xi$

To examine how dependent the proposed MULTIMOORA procedure is on the choice of the Schweizer-Sklar parameter, the values of $p = 3$ and $q = 2$ are held fixed while $\xi$ is varied across a broad range, and the MATLAB implementation is re-executed for each setting. The resulting final utility values and the corresponding rankings are reported in Table 3.

**Table 3**
Sensitivity results for different values of $\xi$

| | Final ranking values | | | | | | |
|---|---|---|---|---|---|---|---|
| $\xi$ | $\Xi_1$ | $\Xi_2$ | $\Xi_3$ | $\Xi_4$ | $\Xi_5$ | $\Xi_6$ | Ranking |
| -1 | 0.2063 | 0.0647 | 0.1696 | -0.1067 | 0.1019 | -0.0321 | $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ |
| -2 | 0.1869 | 0.0647 | 0.1693 | -0.1067 | 0.1220 | -0.0322 | $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ |
| -3 | 0.1675 | 0.0846 | 0.1690 | -0.1067 | 0.1222 | -0.0323 | $Z_3 > Z_1 > Z_5 > Z_2 > Z_6 > Z_4$ |
| -4 | 0.1679 | 0.0848 | 0.1486 | -0.1067 | 0.1426 | -0.0324 | $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ |
| -5 | 0.1682 | 0.0849 | 0.1483 | -0.1066 | 0.1428 | -0.0325 | $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ |
| -10 | 0.1683 | 0.1060 | 0.1276 | -0.0880 | 0.1437 | -0.0515 | $Z_1 > Z_5 > Z_3 > Z_2 > Z_6 > Z_4$ |
| -50 | 0.1666 | 0.1076 | 0.1272 | -0.0873 | 0.1442 | -0.0519 | $Z_1 > Z_5 > Z_3 > Z_2 > Z_6 > Z_4$ |
| -100 | 0.1662 | 0.1077 | 0.1273 | -0.0872 | 0.1441 | -0.0519 | $Z_1 > Z_5 > Z_3 > Z_2 > Z_6 > Z_4$ |
| -150 | 0.1661 | 0.1078 | 0.1273 | -0.0684 | 0.1252 | -0.0519 | $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ |

Table 3 shows that the ranking $Z_1 > Z_3 > Z_5 > Z_2 > Z_6 > Z_4$ holds at $\xi = -1$ and $\xi = -2$, and holds again at $\xi = -4$ and $\xi = -5$. Between these two bands, at $\xi = -3$, $Z_1$ and $Z_3$ momentarily exchange places, so that $Z_3$ rather than $Z_1$ is returned as the top choice. As $\xi$ is pushed further, from $-10$ through $-100$, a second and more lasting reordering takes hold, with $Z_5$ and $Z_3$ trading positions to give $Z_1 > Z_5 > Z_3 > Z_2 > Z_6 > Z_4$, before the original ordering among $Z_3$ and $Z_5$ reasserts itself at $\xi = -150$. Across this whole range, the ordering of $Z_2$, $Z_6$ and $Z_4$ never varies at all, and $Z_4$ remains the last ranked option throughout. Taking all this in combination suggests that the upper middle part of the spectrum - consisting of $Z_1$, $Z_3$ and $Z_5$ - must be where anything that $\xi$ does gets done; these are the three options clustered near enough together on the score continuum such that occasionally a modest change in intensity of aggregation will swap one in to or out of the ranking order, the remainder is never so bothered. Furthermore, $Z_1$ also comes out first out of all six alternatives every time, with the notable exception of $\xi = -3$, at which point it loses out on top spot by a nose, so this can quite reasonably be described as a broadly stable method to the extent that how we order is generally unaffected by how strongly we aggregate, with some slight, localized instability at the very top of the ordering, that involves only a small clustering.

### 6.3.2 Sensitivity regarding Rung Parameters

In this section, we further investigate the robustness of the proposed solution to variations in the parameters $p$ and $q$. Noting that $p, q \in \mathbb{Z}^+$ with $p \geq q$ only enter the model through the fraction $p/q$, this fraction can take any value throughout a continuum as opposed to just step increments of 1, and therefore it is possible that the same admissible region could be achieved by infinite pairs of integers. We maintain the previous case, the one with the smallest ratio whose entire decision matrix $M$ is admissible, from Section 6.1, where $p = 3, q = 2$. Then we test cases where the ratio is gradually increasing (by loosening the fractional exponent) to see when the resulting decision no longer matches the previously determined one.

**Table 4**
Sensitivity results for different values of $p$ and $q$

| $p, q$ | Final ranking values | | | | | | Ranking |
|---|---|---|---|---|---|---|---|
| | $\Xi_1$ | $\Xi_2$ | $\Xi_3$ | $\Xi_4$ | $\Xi_5$ | $\Xi_6$ | |
| 3,2 | 0.1675 | 0.0846 | 0.1690 | -0.1067 | 0.1222 | -0.0323 | $Z_3 > Z_1 > Z_5 > Z_2 > Z_6 > Z_4$ |
| 4,2 | 0.1684 | 0.1431 | 0.1450 | -0.1064 | 0.0815 | -0.0292 | $Z_1 > Z_3 > Z_2 > Z_5 > Z_6 > Z_4$ |
| 5,2 | 0.1687 | 0.1850 | 0.1004 | -0.1048 | 0.0591 | -0.0079 | $Z_2 > Z_1 > Z_3 > Z_5 > Z_6 > Z_4$ |
| 7,3 | 0.1686 | 0.1646 | 0.1216 | -0.1055 | 0.0603 | -0.0086 | $Z_1 > Z_2 > Z_3 > Z_5 > Z_6 > Z_4$ |
| 8,3 | 0.1688 | 0.1857 | 0.0991 | -0.1041 | 0.0189 | 0.0319 | $Z_2 > Z_1 > Z_3 > Z_6 > Z_5 > Z_4$ |
| 10,4 | 0.1687 | 0.1850 | 0.1004 | -0.1048 | 0.0591 | -0.0079 | $Z_2 > Z_1 > Z_3 > Z_5 > Z_6 > Z_4$ |
| 10,6 | 0.1679 | 0.1215 | 0.1491 | -0.1068 | 0.1036 | -0.0312 | $Z_1 > Z_3 > Z_2 > Z_5 > Z_6 > Z_4$ |
| 18,10 | 0.1681 | 0.1421 | 0.1474 | -0.1067 | 0.0829 | -0.0304 | $Z_1 > Z_3 > Z_2 > Z_5 > Z_6 > Z_4$ |

Eight pairs of $(p, q)$, spanning ratios from $p/q = 1.5$ up to $p/q \approx 2.667$, were tested, and the resulting rankings are reported in Table 4. The first observation is that the same pair of integers, e.g., (5, 2) and $(-10, -4)$, which give the same ratio p/q $=$ 2.5, gives the same set of utility values, and hence the same ordering: The second point is that the lower ranked alternative is basically unaffected by the ratio selection: the lowest ranked alternative, $Z_4$, is the least preferred alternative in each and every ratio examined, and in all but one, the next lowest is $Z_6$. The only exception to this is at $p/q \approx 2.667$ where the positions of $Z_6$ and $Z_5$ switch, a change that is too small in magnitude to affect the relative ranking of either. Where the ratio does leave a visible mark is in the upper tier, among $Z_1$, $Z_2$, and $Z_3$: at the tighter ratios, $p/q = 1.5, 1.667, 1.8$, and $2$, the leading position alternates only between $Z_1$ and $Z_3$, with $Z_2$ trailing in third; but once the ratio is relaxed past $2$, at $p/q = 2.333, 2.5$, and $2.667$, $Z_2$ moves up sharply and overtakes both rivals to claim, or nearly claim, the top spot. This pattern is consistent with the loosening of the admissible boundary that a larger ratio brings about, since alternative $Z_2$ carries a comparatively high non-membership component under attribute $\mathbb{A}_1$ that a wider curve discounts less severely than a tighter one does. Overall, the ranking of the proposed methodology is reasonably robust to the choice of rung parameters, particularly in the lower half of the alternative set. At the same time, the upper tier shows a mild, directionally consistent sensitivity that tracks the widening of the fractional boundary as $p/q$ increases.

## 7. Comparison Analysis

This section illustrates the effectiveness of the proposed MULTIMOORA method by comparing its performance with established methodologies [12, 32, 34, 36]. The comparison is conducted using the cloud service provider data set of Section 6.1, and the final outcomes obtained from the existing methods are summarized in Table 5.

**Table 5**
Comparative analysis results

| AOs | Final ranking values | Ranking |
|---|---|---|
| CqROFDWA [32] | $9.5080e^{-05}$, $1.0022e^{-04}$, $3.3023e^{-04}$, $6.1295e^{-06}$, $5.9212e^{-04}$, $9.6718e^{-05}$ | $Z_5 > Z_3 > Z_2 > Z_6 > Z_1 > Z_4$ |
| CqROFDWG [32] | 0.0104, 0.0082, 0.0087, 0.0055, 0.0024, 0.0040 | $Z_1 > Z_3 > Z_2 > Z_4 > Z_6 > Z_5$ |
| q-ROF MULTIMOORA [36] | 0.1779, $6.7596e^{-04}$, 0.1009, -0.1498, 0.2611, 0.0544 | $Z_5 > Z_1 > Z_3 > Z_6 > Z_2 > Z_4$ |
| Cq-ROF MULTIMOORA [34] | 0.1684, 0.1431, 0.1450, -0.1064, 0.0815, -0.0292 | $Z_1 > Z_3 > Z_2 > Z_5 > Z_6 > Z_4$ |
| FOFWA [12] | 0.7755, 0.7633, 0.7266, 0.6946, 0.7537, 0.7711 | $Z_1 > Z_6 > Z_2 > Z_5 > Z_3 > Z_4$ |
| FOFWG [12] | 0.6400, 0.6147, 0.6325, 0.5507, 0.6644, 0.6148 | $Z_5 > Z_1 > Z_3 > Z_6 > Z_2 > Z_4$ |
| Proposed approach | 0.1675, 0.0846, 0.1690, -0.1067, 0.1222, -0.0323 | $Z_3 > Z_1 > Z_5 > Z_2 > Z_6 > Z_4$ |

Three points are worth drawing out of Table 5.

**1).** The two operators proposed by Ali and Yang [32] do not agree with the proposed method, and, more tellingly, do not even agree with each other: circular $q$-rung orthopair fuzzy Dombi weighted average (CqROFDWA) places Z$_5$ first and Z$_4$ last, while circular $q$-rung orthopair fuzzy Dombi weighted geometric (CqROFDWG) places Z$_1$ first and Z$_5$ last, the very alternative CqROFDWA had ranked as the strongest candidate. Neither ordering coincides with the proposed ranking, in which Z$_3$ leads and Z$_4$ trails. The most direct explanation lies in the score function that [32] attaches to a $C_r q$-ROFN, $S(\mathbb{C}) = r^q(\mathcal{M}^q - \mathcal{N}^q)$, a point already raised by Ali et al. [34]: because the score is a *product* of the radius term and the membership spread, and because the radii present in the case study sit in the comparatively narrow band $[0.02, 0.20]$, squaring them under $q = 2$ compresses every score into the $10^{-6}$-$10^{-4}$ range, so that the ranking ends up governed almost entirely by which alternative happens to carry the largest radius rather than by the underlying strength of its membership and non-membership evaluations. The most obvious sign of this coupling is a score of exactly zero whenever $r = 0$ and even though $\mathcal{M}$ might outperform $\mathcal{N}$, Definition 7 ensures that the addition of the radius term does not cancel out this score, but instead is added to it so that a judgment that is made very explicitly, but with a lot of confidence, isn't muted. The second source of disagreement is that [32] relies on a fixed integer rung, whereas the present method is based on the fractional ratio $p/q = 1.5$, which is studied directly in the following point.

**2).** By focusing on the two MULTIMOORA-based comparators, the effect of the fractional exponent is isolated more clearly. The ranking proposed by Riaz et al. [36] is based on plain $q$-ROF data. Consequently, they miss the radius from all evaluations and only observe the center of each evaluation, which is why the ranking in the middle part, Z$_5$ $>$ Z$_1$ $>$ Z$_3$ $>$ Z$_6$ $>$ Z$_2$ $>$ Z$_4$ differs from what is proposed here. The proposed result, on the other hand, is more clearly similar to the result obtained by Ali et al. [34], who still keep the circular structure but fix the rung at an integer value, their ranking being Z$_1$ $>$ Z$_3$ $>$ Z$_2$ $>$ Z$_5$ $>$ Z$_6$ $>$ Z$_4$. The ordering proposed by the method itself is just the same, as shown in Table 4 in Section 6.3.2 when the rung is loosened to an integer ratio as close as possible to $p/q = 2$, which is what [34] is doing. That is, the difference between the proposed approach and the two-off-the-shelf frameworks of Ali et al.'s C$q$-ROF MULTIMOORA is not due to two different unrelated algorithms, but is because a fixed integer rung has been replaced by a continuously tunable one – the exact mechanism that Section 3.1 aims to explain.

**3).** On the other side of the comparison, Zhang et al. [12] fractional orthopair fuzzy operators have no circular part at all, as FOFS has no radius. FOFWA ranks Z$_1$ first and Z$_4$ last and agrees with the proposed method only regarding the worst alternative, but not regarding the best, whereas FOFWG ranks Z$_5$ first. The alternatives whose ordering change the most under FOFWA and FOFWG (two alternatives, Z$_2$ and Z$_6$) are the two candidates whose radius entries are among the largest in the data set; when the radius is dropped from the aggregation, the imprecision that these two alternatives carry is not discounted, and that alone would be sufficient to move them several places in the ordering. Again, as in the opposite direction to point 2), this verifies that the radius term isn't an effectual addition to the model but actually contributes to the overall ranking.

Taken together, these comparisons point to two concrete advantages of the proposed framework over each family of existing operators considered.

1. The CFOFS is the only structure of those compared that is affected by both sources of distortion mentioned above, rounding the fractional judgment to an integer rung as in [34, 36], and discarding the radius entirely as in [12]. The proposed approach will not fail by mistake; indeed, these two restrictions will be eliminated a priori, as discussed in Section 3.1.

2. The score function of Definition 10 adds the radius to the membership spread rather than multiplying by it, so an alternative with a small but non-zero radius is never driven toward a near-zero score the way it is under the formulation of [32]. This keeps the ranking responsive to genuine differences between $\mathcal{M}$ and $\mathcal{N}$ even when the radii are small across the data set, as they are here.

3. Unlike the Dombi weighted operators of [32], which report a single aggregated value per alternative, the proposed method channels the same CFOFN information through three structurally different ranking strategies, the ratio system, the reference point, and the full multiplicative form, and reconciles them through the

Borda rule of Eq. (30)–(33). A single-operator ranking has no internal check on its own conclusions. In contrast, a disagreement between the three MULTIMOORA subsystems is itself informative, and the sensitivity results of Section 6.3.1 show that such disagreement, when it occurs, is confined to a narrow band of the parameter space rather than reflecting an unstable method.

4. The additional Schweizer-Sklar parameter $\xi$ gives the proposed method a degree of adjustability that none of the compared operators possess: $\xi$ can be tuned to the risk posture of a given decision context, as illustrated in Section 6.3.1.

## 8. Conclusions and Future Directions

This article has introduced the CFOFS, a structure built by pairing the continuously tunable exponent ratio $p/q$ of the fractional orthopair fuzzy set with the disk-shaped imprecision region of the C$q$-ROFS, so that a single evaluation can be judged against a boundary that matches the expert's actual confidence rather than the nearest available integer rung, while still absorbing the residual uncertainty surrounding that judgment. The basic theory of the CFOFS, including its score and accuracy functions, its ranking rule, and its fundamental operational laws, was developed in Section 3, and these laws were then reformulated under SS $t$-norms and $t$-conorms to produce the CFOFSSWA and CFOFSSWG operators, whose idempotency, monotonicity, and boundedness properties were established. Building on this aggregation theory, we extended the MULTIMOORA method to the CFOFS environment by embedding the proposed operators into its ratio system, reference point, and full multiplicative subsystems, and reconciling the resulting three rankings through the Borda rule. The framework was applied to a cloud computing service provider selection problem involving six candidate vendors evaluated against four criteria: security and data privacy, scalability and performance, cost-effectiveness, and service reliability and support. The sensitivity analysis showed that the final recommendation is largely stable as the Schweizer-Sklar parameter $\xi$ is varied, with only the closely scored middle tier of alternatives occasionally reordering and the worst alternative remaining unchanged throughout. The complementary analysis of rung parameters showed that identical exponent ratios $p/q$ return identical rankings regardless of which integer pair produces them, that the lower tier of the ranking is essentially insensitive to the choice of rung, and that the small amount of movement observed at the top of the ranking is directionally consistent with the widening of the admissible boundary as $p/q$ increases. The comparative study placed the proposed method alongside existing Dombi-based, plain $q$-ROF, circular $q$-ROF, and fractional orthopair operators, and traced the resulting disagreements to two identifiable sources: the coupling between radius and score in [32] and the loss of either the fractional exponent or the radius term in the remaining comparators, neither of which affects the proposed method by construction.

Several extensions remain open. The weight vector $\varpi$ was fixed by expert judgment; an objective scheme, such as an entropy-based or best-worst-method procedure adapted to the CFOFS environment, would reduce this reliance on subjective weighting. Second, the present model treats all criteria as equally prioritized; developing SS prioritized operators for the CFOFS would extend the framework to settings with an explicit priority ordering among criteria. Third, the radius of an aggregated CFOFN is currently obtained through a simple $\max$ or $\min$ operator, which does not distinguish an aggregation of many similarly imprecise judgments from one dominated by a single outlier; a more appropriate combination mechanism is worth investigating as a replacement.

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## Conflicts of Interest

The authors declare no conflicts of interest.

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