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Probabilistic Shadowed Sets: Persistence and Transience in Fuzzy Uncertainty

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ABSTRACT

Shadowed sets provide a three-way representation of uncertainty by separating positive, negative, and boundary evidence. However, the conventional shadow region is treated as a single homogeneous component, disregarding structural information associated with local variations in fuzziness. This paper introduces *Probabilistic Shadowed Sets (PSS)*, a probabilistic refinement based on the principle that uncertainty possesses both magnitude and structural organization. Starting from a fuzzy membership function, we construct a decomposition into membership, non-membership, persistent uncertainty, and transient uncertainty. The distinction between the two forms of uncertainty is induced by a persistence function derived from local variations in fuzziness, yielding a four-state representation that preserves probability conservation while enriching the descriptive capacity of shadowed-set approximations. We establish fundamental properties, including structural non-invertibility and entropy refinement, and show that the probabilistic representation contains information unavailable in conventional shadowed sets. Experimental studies involving diverse continuous membership functions demonstrate that PSS consistently reveals structural differences that remain indistinguishable under classical shadowed-set approximations. These results position Probabilistic Shadowed Sets as a principled extension of shadowed-set theory for structure-aware uncertainty representation.

1. Introduction

The representation of uncertainty lies at the heart of fuzzy set theory. Since the introduction of fuzzy sets by Zadeh [1], numerous frameworks have been developed to capture different facets of uncertainty, including rough sets [2], intuitionistic fuzzy sets [3], shadowed sets [4], and three-way decision models [5]. Among these, shadowed sets introduced by Pedrycz [4] and applied to interpretation of cluster[6–9], granular computing [10–12], salient region detection in image edge detection [13], occupy a distinctive position by transforming a fuzzy set into a three-region approximation consisting of acceptance (elevated membership), rejection (reduced membership), and uncertainty regions (shadow membership, any value in $(0, 1)$). Rather than eliminating vagueness, shadowed sets concentrate uncertainty into an explicitly identifiable boundary region, thereby providing an interpretable bridge between gradual and discrete representations.

The development of shadowed sets has largely focused on determining optimal boundaries between these regions. Various principles have been proposed, including minimum approximation error [14, 15], uncertainty balance [6, 16], average uncertainty minimization [11], nearest quota of fuzziness [17], entropy-based optimization [18, 19], gradual-number formulations [20], and game-theoretic strategies [21, 22]. Other studies have extended the framework to five-region approximations [23], multigranularity

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shadowed structures [24], generalized fuzzy environments [25, 26] and fuzziness-driven three-way decision models [27, 28]. These developments have substantially improved the construction and applicability of shadowed set approximations.

Despite these advances, a fundamental question remains largely unexplored. *Once uncertainty has been condensed into the shadow region, is that uncertainty itself structurally uniform?*

Classical shadowed sets implicitly answer this question in the affirmative. All elements assigned to the shadow region are represented through a single undifferentiated state regardless of how the underlying uncertainty is organized. Consequently, two elements may possess identical degrees of fuzziness while exhibiting markedly different local uncertainty behavior. One may belong to a stable zone in which uncertainty persists over an extended neighborhood, whereas another may arise from a rapidly changing transition between acceptance and rejection. Within existing shadowed-set models, these distinct situations become indistinguishable.

This observation suggests that uncertainty possesses two complementary attributes. The first attribute is its magnitude, describing how uncertain an element is. The second attribute is its structural organization, describing how that uncertainty is distributed locally within the underlying fuzzy representation. Conventional shadowed sets capture the former but remain insensitive to the latter. As a result, potentially informative structural characteristics are lost when the shadow region is represented as a single homogeneous component.

The objective of this paper is to develop a representation capable of preserving this latent structural information. Taking insights from Yager [29], we begin by introducing the equilibrium deviation

$$\delta(x) = 2\mu_A(x) - 1,$$

which quantifies displacement from the point of maximum uncertainty. From this quantity we take the fuzziness degree as

$$\nu(x) = 1 - |\delta(x)|,$$

which measures the local intensity of uncertainty. We then investigate how uncertainty varies across the domain through the gradient of $\nu(x)$. This leads to the concepts of intrinsic persistence and intrinsic transience, which distinguish uncertainty that remains locally stable from uncertainty associated with rapid transitions. The resulting decomposition induces a probability quadruple

$$\Pi(x) = \big(p_M(x), p_P(x), p_T(x), p_N(x)\big),$$

representing membership, persistent uncertainty, transient uncertainty, and non-membership, respectively.

The proposed framework, termed *Probabilistic Shadowed Sets (PSS)*, extends the classical three-way representation without introducing external tuning parameters. The construction preserves probability conservation, admits a natural balance-induced threshold mechanism, and retains linear computational complexity. More importantly, it enriches the descriptive capacity of shadowed set approximations by revealing structural distinctions that remain hidden within conventional models.

The theoretical development is supported by a collection of validation studies. We establish a structural non-invertibility property showing that the probabilistic representation constitutes a strict refinement of conventional fuzzy membership. An entropy refinement result demonstrates that the probabilistic decomposition preserves and enriches uncertainty information. Furthermore, structural discrimination experiments involving continuous membership functions reveal that probabilistic shadowed sets consistently identify differences in uncertainty organization that classical shadowed approximations either suppress or substantially attenuate. Additional locality analyses confirm that the proposed representation captures meaningful structural variability while preserving coherent uncertainty behavior.

The contributions of this work are summarized as follows:

- We identify and formalize the notion of structural organization within the shadow region of a fuzzy set.

- We introduce intrinsic persistence and transience as first-principles descriptors derived from local variations of fuzziness.
- We develop Probabilistic Shadowed Sets, a parameter-free probabilistic refinement that transforms the shadow region into a structured uncertainty representation.
- We establish fundamental theoretical properties including structural non-invertibility, probability conservation, and entropy refinement.
- We provide experimental evidence that demonstrates that the proposed framework reveals structural distinctions that remain inaccessible to conventional shadowed-set representations.

The remainder of this paper is organized as follows. Section 2 develops the equilibrium geometry of fuzzy membership. Section 3 derives the intrinsic probabilistic representation. Section 4 formulates Probabilistic Shadowed Sets. Section 5 presents theoretical validation studies. Section 6 concludes the paper.

2. Equilibrium Geometry of Fuzzy Membership

The term *equilibrium* in this paper is used in a specific technical sense. It refers to the point at which the competing tendencies toward membership and non-membership are perfectly balanced. For a given element x, this occurs precisely when the membership degree $\mu_A(x)$ equals its complement $1 - \mu_A(x)$, i.e., when $\mu_A(x) = 0.5$. At this point, neither tendency dominates, and the local uncertainty is maximal.

The structure of this equilibrium — whether it is sustained over a region or encountered only transiently — determines the nature of uncertainty and forms the foundation of the Probabilistic Shadowed Set framework.

2.1 Observational Foundation

Consider a fuzzy set A defined on a universe U with membership function $\mu_A : U \to [0, 1]$. The grade $\mu_A(x)$ represents the degree to which element x belongs to the concept represented by A. This function is not an arbitrary mathematical construction; it arises from empirical observations, expert knowledge, or learning processes. Consequently, its structure encodes information about the underlying phenomenon.

Two fundamental observations about any membership function are immediate:

1. For each $x \in U$, the membership grade $\mu_A(x)$ and its complement $1 - \mu_A(x)$ form a competitive pair: an increase in one decreases the other. They represent, respectively, the tendencies toward membership and non-membership.
2. The degree of uncertainty experienced at x is maximal when these two competitive tendencies are perfectly balanced, i.e., when $\mu_A(x) = 1 - \mu_A(x)$. In contrast, when $\mu_A(x)$ approaches 0 or 1, the element becomes decisively classified and uncertainty vanishes.

These observations suggest that uncertainty is not an intrinsic property of the element x alone, but rather emerges from the interplay between membership and non-membership tendencies. The structure of this interplay reveals the nature of uncertainty.

2.2 Equilibrium Deviation and Fuzziness

Let us formalize the competitive relationship between membership and non-membership.

Definition 2.1 (Equilibrium Deviation)**.** The equilibrium deviation at $x \in U$ is defined as

$$\delta(x) = \mu_A(x) - (1 - \mu_A(x)) = 2\mu_A(x) - 1. \tag{1}$$

The equilibrium deviation $\delta(x) \in [-1, 1]$ measures the signed imbalance between membership and non-membership tendencies. The sign of $\delta(x)$ identifies the dominant tendency: positive values indicate membership dominance, negative values indicate non-membership dominance. The magnitude $|\delta(x)|$ measures the strength of this dominance.

When $\delta(x) = 0$, the two tendencies are perfectly balanced. This occurs precisely when $\mu_A(x) = 0.5$. We refer to this state as *equilibrium*.

Definition 2.2 (Fuzziness Degree)**.** The fuzziness degree at $x \in U$ is defined as

$$\nu(x) = 1 - |\delta(x)| = 1 - |2\mu_A(x) - 1|. \tag{2}$$

The fuzziness degree $\nu(x) \in [0, 1]$ measures the local uncertainty experienced at x. It attains its maximum value $\nu(x) = 1$ precisely at equilibrium ($\mu_A(x) = 0.5$), and its minimum value $\nu(x) = 0$ at complete commitment ($\mu_A(x) \in \{0, 1\}$). This quantity is entirely determined by the membership function and requires no external parameters.

Remark 2.3. The fuzziness degree $\nu(x)$ coincides with the membership function of the fuzziness set ϕ_A introduced by Tahayori et al. [20]. However, here it emerges directly from the observed competitive relationship between membership and non-membership, rather than being postulated as a measure.

2.3 The Geometry of Fuzziness

The function $\nu : U \to [0, 1]$ describes the distribution of uncertainty across the universe. Its geometry encodes structural information about how uncertainty is generated.

Definition 2.4 (Fuzziness Set)**.** The fuzziness set of A is defined as

$$\phi_A = \{(x, \nu(x)) : x \in U\}. \tag{3}$$

The fuzziness set ϕ_A is a fuzzy set whose membership function is precisely the fuzziness degree. Its support is the region where $\nu(x) > 0$, corresponding to $\mu_A(x) \in (0, 1)$. The points of maximum fuzziness satisfy $\nu(x) = 1$, i.e., $\mu_A(x) = 0.5$.

To analyze the geometry of ν, we examine its local behavior. A fundamental question arises:

How rapidly does fuzziness change as we move through the universe?

This question is of central importance because the rate of change of fuzziness reveals whether uncertainty is sustained over a region or occurs only transiently at a boundary.

To answer this question, we examine the relationship between fuzziness and membership. Recall that

$$\nu(x) = 1 - |2\mu_A(x) - 1|. \tag{4}$$

This function depends on $\mu_A(x)$ through the absolute value. To determine how ν changes with respect to μ_A, we consider two cases.

1. *When $\mu_A(x) \le 0.5$:* Here, $2\mu_A(x) - 1 \le 0$, so $|2\mu_A(x) - 1| = 1 - 2\mu_A(x)$. Thus,

$$\nu(x) = 1 - (1 - 2\mu_A(x)) = 2\mu_A(x). \tag{5}$$

Consequently,

$$\frac{d\nu}{d\mu_A} = 2. \tag{6}$$

This means that on the left side of equilibrium, fuzziness increases with membership.

2. *When $\mu_A(x) \ge 0.5$:* Here, $2\mu_A(x) - 1 \ge 0$, so $|2\mu_A(x) - 1| = 2\mu_A(x) - 1$. Thus,

$$\nu(x) = 1 - (2\mu_A(x) - 1) = 2(1 - \mu_A(x)). \tag{7}$$

Consequently,

$$\frac{d\nu}{d\mu_A} = -2. \tag{8}$$

This means that on the right side of equilibrium, fuzziness decreases with membership.

These two cases can be unified into a single expression. Observe that the derivative of ν with respect to μ_A is $+2$ when $\mu_A < 0.5$ and -2 when $\mu_A > 0.5$. This is exactly:

$$\frac{d\nu}{d\mu_A} = -2 \cdot \mathrm{sgn}(2\mu_A(x) - 1), \tag{9}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function. Since $\delta(x) = 2\mu_A(x) - 1$ is the equilibrium deviation, we have:

$$\frac{d\nu}{d\mu_A} = -2 \cdot \mathrm{sgn}(\delta(x)). \tag{10}$$

Now, applying the chain rule to obtain the spatial gradient, we get:

$$\nabla\nu(x) = \frac{d\nu}{d\mu_A} \cdot \nabla\mu_A(x) = -2 \cdot \mathrm{sgn}(\delta(x)) \cdot \nabla\mu_A(x). \tag{11}$$

This derivation leads to the following definition.

Definition 2.5 (Fuzziness Gradient)**.** For a differentiable membership function, the gradient of fuzziness at $x \in U$ is defined as

$$\nabla\nu(x) = -2 \cdot \mathrm{sgn}(2\mu_A(x) - 1) \cdot \nabla\mu_A(x), \tag{12}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function.

The gradient $\nabla\nu(x)$ measures how rapidly fuzziness changes in the neighborhood of x. Its magnitude $|\nabla\nu(x)|$ indicates the rate of transition between uncertainty and commitment. Crucially, this definition is not an external imposition; it is an inevitable consequence of the relationship between fuzziness and membership.

Two qualitatively distinct situations arise naturally from the geometry of ν:

1. *Sustained Fuzziness:* When $|\nabla\nu(x)| \approx 0$ over a region, the fuzziness remains essentially unchanged. The equilibrium state is maintained over an extended interval.

2. *Transitional Fuzziness:* When $|\nabla\nu(x)| \gg 0$ at a point, the fuzziness changes rapidly. The equilibrium state is encountered only momentarily before commitment is recovered.

These observations lead to an intrinsic characterization of uncertainty persistence.

2.4 *Intrinsic Persistence and Transience*

The preceding analysis of the fuzziness gradient reveals that uncertainty exhibits fundamentally different local behaviors. When $|\nabla\nu(x)| \approx 0$, fuzziness is locally constant and the equilibrium state is sustained. When $|\nabla\nu(x)| \gg 0$, fuzziness changes rapidly and equilibrium is encountered only transiently.

To capture this distinction formally, we introduce the concept of persistence.

2.4.1 *Behavioral Characterization of Persistence*

Consider the fuzziness field $\nu : U \to [0, 1]$. For an element $x \in U$, the degree to which uncertainty persists in a neighborhood of x is determined by how slowly ν varies in that neighborhood.

Definition 2.6 (Behavioral Characterization of Persistence)**.** *Persistence* is the degree to which the fuzziness state at a point is maintained across its spatial neighborhood. Equivalently, it is the likelihood that a small displacement from x leaves the uncertainty state essentially unchanged.

This characterization leads to two fundamental observations:

1. When ν changes slowly, uncertainty at x resembles uncertainty at nearby points. The uncertainty state is sustained — it *persists*.

2. When ν changes rapidly, uncertainty at x differs substantially from its surroundings. The uncertainty state is localized — it does *not persist*.

These observations establish that persistence is fundamentally a measure of local stability of the fuzziness field. More formally:

$$\text{Persistence at } x \propto \frac{1}{\text{rate of change of } \nu \text{ at } x}. \tag{13}$$

This relationship yields three necessary conditions that any persistence function $P(x)$ must satisfy.

2.4.2 Axioms for Persistence

1. *Monotonicity.* Persistence decreases monotonically with increasing local variation of fuzziness. Formally, there exists a strictly decreasing function $\Phi : \mathbb{R}_{\geq 0} \to (0, 1]$ such that

$$P(x) = \Phi(|\nabla \nu(x)|).$$

2. *Boundary Conditions.* At the extremes of behavior:

$$\Phi(0) = 1, \qquad \lim_{g \to \infty} \Phi(g) = 0.$$

When fuzziness is perfectly flat ($\nabla \nu = 0$), uncertainty persists perfectly ($P = 1$). When fuzziness changes infinitely rapidly ($|\nabla \nu| \to \infty$), uncertainty does not persist at all ($P = 0$).

3. *Simplicity.* Among all functions satisfying Monotonicity and the Boundary Conditions, the persistence function should be the simplest rational form, introducing no unnecessary parameters.

These axioms follow directly from the behavioral characterization of persistence as local stability of the fuzziness field.

2.4.3 Derivation of Persistence

Let $g = |\nabla \nu(x)| \geq 0$. By Axiom 1, $P(x) = \Phi(g)$ for some strictly decreasing function Φ. By Axiom 2, $\Phi(0) = 1$ and $\lim_{g \to \infty} \Phi(g) = 0$. By Axiom 3, among all rational functions satisfying these constraints, the simplest is:

$$\Phi(g) = \frac{1}{1 + \alpha g}, \quad \alpha > 0.$$

The boundary condition $\Phi(0) = 1$ is automatically satisfied. The constant α determines the rate of decay. Since $|\nabla \nu|$ has units of inverse length, α has units of length. Normalizing $\alpha = 1$ defines the unit of gradient measurement, yielding:

$$P(x) = \frac{1}{1 + |\nabla \nu(x)|}. \tag{14}$$

Thus, the persistence function is the unique minimal rational form consistent with the three axioms.

Definition 2.7 (Intrinsic Persistence). The persistence of fuzziness at $x \in U$ is defined as

$$P(x) = \frac{1}{1 + |\nabla \nu(x)|}. \tag{15}$$

This definition possesses the following natural properties:

1. $P(x) \in (0, 1]$ for all $x \in U$.

2. $P(x) = 1$ if and only if $\nabla \nu(x) = 0$ (perfect persistence).

3. $P(x) \to 0$ as $|\nabla\nu(x)| \to \infty$ (zero persistence).
4. $P(x)$ is entirely determined by the geometry of the membership function; no external parameters are introduced.

Definition 2.8 (Intrinsic Transience). The transience of fuzziness at $x \in U$ is defined as

$$T(x) = 1 - P(x) = \frac{|\nabla\nu(x)|}{1 + |\nabla\nu(x)|}. \tag{16}$$

The transience $T(x) \in [0, 1)$ measures the degree to which fuzziness is transitional rather than sustained. When $T(x) \approx 0$, the fuzziness is highly persistent. When $T(x) \approx 1$, the fuzziness is highly transient.

Remark 2.9. The quantities $P(x)$ and $T(x)$ are complementary: $P(x) + T(x) = 1$. They form a natural decomposition of the local behavior of fuzziness into sustained and transitional components, induced entirely by the geometry of the membership function. The ratio

$$\frac{P(x)}{T(x)} = \frac{1}{|\nabla\nu(x)|}$$

directly expresses the relative prevalence of persistence over transience. When the fuzziness gradient is small, persistence dominates; when the gradient is large, transience dominates.

Remark 2.10. The persistence function is not arbitrary. It is the unique minimal rational function satisfying three natural axioms derived directly from the behavioral characterization of persistence as local stability of the fuzziness field. Alternatives such as $e^{-|\nabla\nu|}$, $1/(1 + |\nabla\nu|^2)$, or $1/(1 + \sqrt{|\nabla\nu|})$ either introduce unnecessary parameters or violate the simplicity axiom. The linear form $1/(1 + |\nabla\nu|)$ is the simplest rational function consistent with the axioms.

2.5 Structural Dichotomy of Equilibrium

The equilibrium region is naturally characterized by high fuzziness, i.e., $\nu(x) \approx 1$. However, the nature of equilibrium differs fundamentally depending on whether persistence or transience dominates.

Definition 2.11 (Persistent Equilibrium). An element $x \in U$ exhibits *persistent equilibrium* if

$$\nu(x) \text{ is high and } P(x) \text{ is high.} \tag{17}$$

The collection of such elements is denoted by

$$\mathcal{E}_P = \{x \in U : \nu(x) \geq \eta \text{ and } P(x) \geq \tau\}, \tag{18}$$

where η and τ are structural thresholds to be determined.

Definition 2.12 (Transient Equilibrium). An element $x \in U$ exhibits *transient equilibrium* if

$$\nu(x) \text{ is high and } T(x) \text{ is high.} \tag{19}$$

The collection of such elements is denoted by

$$\mathcal{E}_T = \{x \in U : \nu(x) \geq \eta \text{ and } T(x) > \tau\}. \tag{20}$$

Equivalently, since $T(x) = 1 - P(x)$, the transient equilibrium condition can be expressed as $\nu(x) \geq \eta$ and $P(x) < \tau$.

Theorem 2.13 (Structural Decomposition). *The equilibrium region admits a unique decomposition into persistent and transient components:*

$$\mathcal{E} = \mathcal{E}_P \cup \mathcal{E}_T, \quad \mathcal{E}_P \cap \mathcal{E}_T = \varnothing. \tag{21}$$

Proof. The decomposition follows directly from the definitions of $\mathcal{E}_P$ and $\mathcal{E}_T$. Since $P(x) \geq \tau$ and $P(x) < \tau$ are mutually exclusive conditions for any x, the sets are disjoint. Their union is the set of elements satisfying $\nu(x) \geq \eta$, which is precisely the equilibrium region. □

2.6 *Committed Regions*

Beyond the equilibrium region, elements exhibit varying degrees of commitment. The committed regions are those where $\nu(x) < \eta$, i.e., where fuzziness is sufficiently low.

Definition 2.14 (Membership Region)**.** The membership region is defined as

$$M = \{x \in U : \nu(x) < \eta \text{ and } \delta(x) > 0\}. \tag{22}$$

Elements in M exhibit dominance of membership over non-membership, with sufficiently low fuzziness to warrant commitment.

Definition 2.15 (Non-Membership Region)**.** The non-membership region is defined as

$$N = \{x \in U : \nu(x) < \eta \text{ and } \delta(x) < 0\}. \tag{23}$$

Elements in N exhibit dominance of non-membership over membership, with sufficiently low fuzziness to warrant commitment.

2.7 *The Four Structural States*

The preceding analysis yields a complete decomposition of the universe into four structural states:

$$U = M \cup \mathcal{E}_P \cup \mathcal{E}_T \cup N. \tag{24}$$

These states are:

1. *Membership* (M): Decisive commitment toward membership.
2. *Persistent Equilibrium* ($\mathcal{E}_P$): Sustained uncertainty over a region.
3. *Transient Equilibrium* ($\mathcal{E}_T$): Localized uncertainty at a transition.
4. *Non-Membership* (N): Decisive commitment toward non-membership.

Remark 2.16. The present theory deliberately refrains from interpreting $\mathcal{E}_P$ as ”ignorance” and $\mathcal{E}_T$ as ”conflict” at this stage. Such interpretations require additional evidence structures beyond the membership function alone. The structural dichotomy is purely geometric: it distinguishes sustained equilibrium from transitional equilibrium based on the behavior of ν and its gradient.

Theorem 2.17 (Structural Completeness)**.** *The four structural states M, $\mathcal{E}_P$, $\mathcal{E}_T$, and N form a complete partition of U that is entirely induced by the geometry of μ_A.*

Proof. For any $x \in U$, either $\nu(x) \geq \eta$ (equilibrium) or $\nu(x) < \eta$ (committed). If $\nu(x) \geq \eta$, then either $P(x) \geq \tau$ (persistent) or $P(x) < \tau$ (transient). If $\nu(x) < \eta$, then either $\delta(x) > 0$ (membership) or $\delta(x) < 0$ (non-membership). Thus every x belongs to exactly one of the four states. No element is left unclassified. □

3. Intrinsic Probabilistic Representation

3.1 *Observational Foundation: Uncertainty as a Compositional Structure*

In Section 2, we established that each element $x \in U$ possesses three fundamental uncertainty descriptors:

1. The fuzziness degree $\nu(x) \in [0, 1]$, measuring the magnitude of uncertainty.
2. The persistence $P(x) \in (0, 1]$, measuring the sustained nature of uncertainty.
3. The transience $T(x) \in [0, 1)$, measuring the transitional nature of uncertainty.

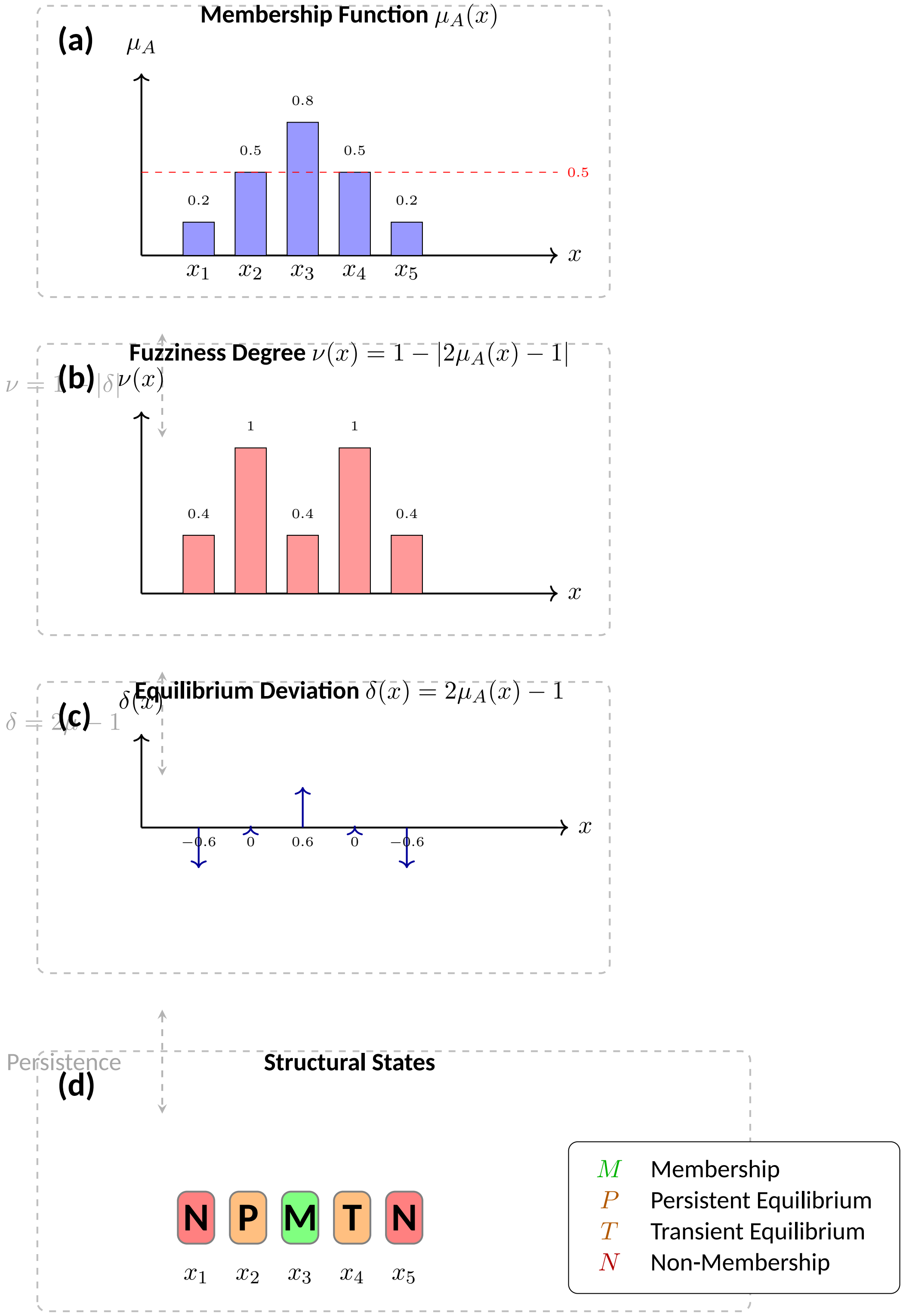

Fig. 1. Geometry of uncertainty: From membership to structural states. (a) A fuzzy set μ_A on five elements $x_1, \ldots, x_5$. (b) Fuzziness degree $\nu(x) = 1 - |2\mu_A(x) - 1|$, maximum at equilibrium $\mu = 0.5$. (c) Equilibrium deviation $\delta(x) = 2\mu_A(x) - 1$; sign indicates direction of commitment. (d) The four structural states: M (membership), P (persistent equilibrium), T (transient equilibrium), N (non-membership)

These three quantities satisfy $\nu(x) \in [0,1]$ and $P(x) + T(x) = 1$. They are entirely determined by the geometry of the membership function.

A fundamental observation now arises:

The uncertainty of an element is not a monolithic quantity but a compositional structure.

The fuzziness degree $\nu(x)$ represents the total uncertainty mass at x. This mass is distributed across the different modes of uncertainty: persistent equilibrium, transient equilibrium, and commitment. The distribution is determined by the local geometry of ν.

This observation leads naturally to a probabilistic interpretation: the total uncertainty mass $\nu(x)$ should be allocated among the possible structural states in proportion to their geometric evidence.

3.2 Intrinsic Probabilistic Structure

The preceding analysis has established that each element $x \in U$ possesses three fundamental descriptors:

1. The equilibrium deviation magnitude $|\delta(x)|$, measuring the degree of commitment.
2. The fuzziness degree $\nu(x) = 1 - |\delta(x)|$, measuring the total uncertainty mass.
3. The persistence $P(x)$ and transience $T(x) = 1 - P(x)$, measuring how uncertainty behaves locally.

These descriptors naturally induce a probabilistic structure. The key observation is that the total uncertainty mass $\nu(x)$ must be allocated between persistent and transient equilibrium in proportion to their local prevalence, as measured by $P(x)$ and $T(x)$, respectively. This yields the following evidence components.

Consider the three fundamental uncertainty components at x:

1. *Commitment Evidence:* The degree to which commitment dominates. This is measured by the equilibrium deviation magnitude $|\delta(x)|$. When $|\delta(x)|$ is large, the element is far from equilibrium and commitment is strong. When $|\delta(x)|$ is small, commitment is weak.
2. *Persistent Equilibrium Evidence:* The degree to which uncertainty is sustained. This is measured by the product $\nu(x)P(x)$. High fuzziness combined with high persistence indicates sustained equilibrium.
3. *Transient Equilibrium Evidence:* The degree to which uncertainty is transitional. This is measured by the product $\nu(x)T(x)$. High fuzziness combined with high transience indicates transient equilibrium.

These three components form a complete evidence structure:

$$\text{Total Evidence} = |\delta(x)| + \nu(x)P(x) + \nu(x)T(x). \tag{25}$$

However, note that:

$$\begin{aligned}
|\delta(x)| + \nu(x)P(x) + \nu(x)T(x) &= |\delta(x)| + \nu(x)(P(x) + T(x)) && (26)\\
&= |\delta(x)| + \nu(x) && (27)\\
&= |\delta(x)| + (1 - |\delta(x)|) && (28)\\
&= 1. && (29)
\end{aligned}$$

Thus the total evidence is identically 1 for all $x \in U$.

This is a remarkable result. The three components:

$$E_C(x) = |\delta(x)|, \tag{30}$$
$$E_P(x) = \nu(x)P(x), \tag{31}$$
$$E_T(x) = \nu(x)T(x), \tag{32}$$

form a partition of unity:

$$E_C(x) + E_P(x) + E_T(x) = 1. \tag{33}$$

Each component represents the intrinsic evidence for a particular structural state. Since these evidence components are non-negative and sum to 1, they constitute a probability distribution over the three fundamental states: commitment (C), persistent equilibrium (P), and transient equilibrium (T).

This observation motivates the following definitions.

Definition 3.1 (Intrinsic Commitment Probability)**.** The intrinsic probability of commitment at $x \in U$ is defined as

$$p_C(x) = |\delta(x)| = |2\mu_A(x) - 1|. \tag{34}$$

The quantity $p_C(x)$ measures the degree to which the element is committed toward either membership or non-membership. It satisfies the following properties:

1. $p_C(x) \in [0, 1]$ for all $x \in U$.
2. $p_C(x) = 0$ if and only if $\mu_A(x) = 0.5$, i.e., at perfect equilibrium.
3. $p_C(x) = 1$ if and only if $\mu_A(x) \in \{0, 1\}$, i.e., at complete commitment.

Remark 3.2. The terminology "probability" is justified as follows. The quantity $p_C(x)$ satisfies the fundamental axioms of probability:

1. *Non-negativity:* $p_C(x) \geq 0$ for all $x \in U$.
2. *Normalization:* $p_C(x) + \nu(x) = 1$, where $\nu(x)$ is the complementary fuzziness degree.
3. *Additivity:* For mutually exclusive events, $p_C(x) = p_M(x) + p_N(x)$, where $p_M(x)$ and $p_N(x)$ are the intrinsic probabilities of membership and non-membership, respectively.

Thus $p_C(x)$ is not an arbitrary measure but a genuine probability induced by the geometry of the membership function.

The complementarity between commitment and fuzziness is fundamental to the theory:

$$p_C(x) + \nu(x) = |\delta(x)| + (1 - |\delta(x)|) = 1. \tag{35}$$

When commitment probability is high, fuzziness is low, and vice versa. At perfect equilibrium ($\mu_A(x) = 0.5$), commitment probability is zero and fuzziness is maximal.

Definition 3.3 (Intrinsic Persistent Equilibrium Probability)**.** The intrinsic probability of persistent equilibrium at $x \in U$ is defined as

$$p_P(x) = \nu(x)P(x) = (1 - |\delta(x)|) \cdot \frac{1}{1 + |\nabla \nu(x)|}. \tag{36}$$

This probability represents the degree to which uncertainty is sustained. When $p_P(x)$ is high, the element exhibits persistent equilibrium: fuzziness is locally constant and the equilibrium state is maintained over an extended region.

Definition 3.4 (Intrinsic Transient Equilibrium Probability)**.** The intrinsic probability of transient equilibrium at $x \in U$ is defined as

$$p_T(x) = \nu(x)T(x) = (1 - |\delta(x)|) \cdot \frac{|\nabla\nu(x)|}{1 + |\nabla\nu(x)|}. \tag{37}$$

This probability represents the degree to which uncertainty is transitional. When $p_T(x)$ is high, the element exhibits transient equilibrium: fuzziness changes rapidly and equilibrium is encountered only momentarily.

Remark 3.5. The products $\nu(x)P(x)$ and $\nu(x)T(x)$ arise naturally from the evidence structure established above. The total fuzziness $\nu(x)$ is allocated between persistent and transient modes in proportion to $P(x)$ and $T(x)$, respectively. No external parameters are introduced; the probabilities are entirely induced by the geometry of the membership function.

The commitment probability naturally decomposes into two directional components corresponding to the sign of the equilibrium deviation:

Definition 3.6 (Intrinsic Membership and Non-Membership Probabilities)**.** The intrinsic probability of membership at $x \in U$ is defined as

$$p_M(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)>0\}}, \tag{38}$$

and the intrinsic probability of non-membership at $x \in U$ is defined as

$$p_N(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)<0\}}. \tag{39}$$

These directional probabilities satisfy:

$$p_M(x) + p_N(x) = p_C(x) = |\delta(x)|. \tag{40}$$

When $\delta(x) > 0$, the element is committed toward membership $(p_M > 0, p_N = 0)$. When $\delta(x) < 0$, the element is committed toward non-membership $(p_N > 0, p_M = 0)$. When $\delta(x) = 0$, both directional probabilities are zero, indicating perfect equilibrium.

Remark 3.7. The probabilities $p_M(x)$ and $p_N(x)$ are not arbitrary assignments but emerge directly from the sign and magnitude of the equilibrium deviation. This ensures that the probabilistic representation is entirely induced by the geometry of the membership function.

Theorem 3.8 (Complete Probability Conservation)**.** *For every $x \in U$, the four intrinsic probabilities satisfy*

$$p_M(x) + p_P(x) + p_T(x) + p_N(x) = 1. \tag{41}$$

Proof. We first establish conservation for the three-state distribution:

$$\begin{aligned}
p_C(x) + p_P(x) + p_T(x) &= |\delta(x)| + \nu(x)P(x) + \nu(x)T(x) &(42)\\
&= |\delta(x)| + \nu(x)(P(x) + T(x)) &(43)\\
&= |\delta(x)| + \nu(x) &(44)\\
&= |\delta(x)| + (1 - |\delta(x)|) &(45)\\
&= 1. &(46)
\end{aligned}$$

Since $p_M(x) + p_N(x) = p_C(x)$, it follows that:

$$\begin{aligned}
p_M(x) + p_P(x) + p_T(x) + p_N(x) &= (p_M(x) + p_N(x)) + p_P(x) + p_T(x) &(47)\\
&= p_C(x) + p_P(x) + p_T(x) &(48)\\
&= 1. &(49)
\end{aligned}$$

Thus the four probabilities form a complete probability distribution over the four structural states: membership (M), persistent equilibrium (P), transient equilibrium (T), and non-membership (N). □

Remark 3.9. The four-state distribution is the Probabilistic Shadowed Set representation:

$$\Pi(x) = (p_M(x), p_P(x), p_T(x), p_N(x)) \,. \tag{50}$$

Each component is entirely determined by the geometry of the membership function. The distribution satisfies conservation, non-negativity, and the fundamental property that $p_P(x) + p_T(x) = \nu(x)$ — the total equilibrium probability equals the fuzziness degree. This establishes the complete probabilistic structure induced by the geometry of fuzzy membership.

3.3 Properties of the Intrinsic Probabilities

We now establish several fundamental properties of the intrinsic probabilities.

Theorem 3.10 (Equilibrium Maximality). *At perfect equilibrium $\mu_A(x) = 0.5$, we have $\delta(x) = 0$, so $p_M(x) = p_N(x) = 0$. The probabilities reduce to:*

$$p_P(x) = P(x), \quad p_T(x) = T(x). \tag{51}$$

Thus the equilibrium probabilities are determined entirely by the local geometry of fuzziness.

Proof. At perfect equilibrium, $\mu_A(x) = 0.5$. By definition of equilibrium deviation, $\delta(x) = 2(0.5)-1 = 0$. Therefore $|\delta(x)| = 0$, and consequently $\nu(x) = 1 - |\delta(x)| = 1$.

For the directional commitment probabilities:

$$p_M(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)>0\}} = 0 \cdot \mathbf{1}_{\{0>0\}} = 0, \tag{52}$$

$$p_N(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)<0\}} = 0 \cdot \mathbf{1}_{\{0<0\}} = 0. \tag{53}$$

For the equilibrium probabilities:

$$p_P(x) = \nu(x)P(x) = 1 \cdot P(x) = P(x), \tag{54}$$

$$p_T(x) = \nu(x)T(x) = 1 \cdot T(x) = T(x). \tag{55}$$

Thus the theorem is established. □

Theorem 3.11 (Commitment Dominance). *As $|\delta(x)| \to 1$, we have $\nu(x) \to 0$, so $p_P(x) \to 0$ and $p_T(x) \to 0$. The probabilities reduce to:*

$$p_M(x) \to \mathbf{1}_{\{\delta(x)>0\}}, \quad p_N(x) \to \mathbf{1}_{\{\delta(x)<0\}}. \tag{56}$$

Thus complete commitment yields deterministic probabilities.

Proof. As $|\delta(x)| \to 1$, by definition of fuzziness degree:

$$\nu(x) = 1 - |\delta(x)| \to 0. \tag{57}$$

Since $p_P(x) = \nu(x)P(x)$ and $P(x) \in (0, 1]$, we have:

$$0 \leq p_P(x) \leq \nu(x) \to 0, \tag{58}$$

so $p_P(x) \to 0$ by the squeeze theorem.

Similarly, since $p_T(x) = \nu(x)T(x)$ and $T(x) \in [0, 1)$, we have:

$$0 \leq p_T(x) \leq \nu(x) \to 0, \tag{59}$$

so $p_T(x) \to 0$.

For the directional probabilities:

$$p_M(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)>0\}} \to 1 \cdot \mathbf{1}_{\{\delta(x)>0\}}. \tag{60}$$

Similarly,

$$p_N(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)<0\}} \to 1 \cdot \mathbf{1}_{\{\delta(x)<0\}}. \tag{61}$$

Thus the theorem is established. □

Theorem 3.12 (Persistence Dominance). *When $|\nabla\nu(x)| \to 0$, we have $P(x) \to 1$ and $T(x) \to 0$. The equilibrium probabilities become:*

$$p_P(x) \to \nu(x), \quad p_T(x) \to 0. \tag{62}$$

Thus persistent equilibrium dominates when fuzziness is flat.

Proof. By definition, $P(x) = 1/(1 + |\nabla\nu(x)|)$. As $|\nabla\nu(x)| \to 0$, we have $P(x) \to 1$, $T(x) = 1 - P(x) \to 0$, and hence $p_P(x) = \nu(x)P(x) \to \nu(x)$ and $p_T(x) = \nu(x)T(x) \to 0$. Thus the theorem is established. □

Theorem 3.13 (Transience Dominance). *When $|\nabla\nu(x)| \to \infty$, we have $P(x) \to 0$ and $T(x) \to 1$. The equilibrium probabilities become:*

$$p_P(x) \to 0, \quad p_T(x) \to \nu(x). \tag{63}$$

Thus transient equilibrium dominates when fuzziness changes rapidly.

Proof. By definition, $P(x) = 1/(1 + |\nabla\nu(x)|)$. As $|\nabla\nu(x)| \to \infty$, $P(x) \to 0$, $T(x) = 1 - P(x) \to 1$, so $p_P(x) = \nu(x)P(x) \to 0$ and $p_T(x) = \nu(x)T(x) \to \nu(x)$. Thus the theorem is established. □

Theorem 3.14 (Continuity). *The probabilities $p_M(x)$, $p_P(x)$, $p_T(x)$, and $p_N(x)$ are continuous functions of μ_A whenever μ_A is continuously differentiable.*

Proof. Each probability is a composition of continuous functions:

First, $\delta(x) = 2\mu_A(x) - 1$ is continuous since μ_A is continuous.

The absolute value function $|\cdot|$ is continuous, so $|\delta(x)|$ is continuous.

The fuzziness degree $\nu(x) = 1 - |\delta(x)|$ is continuous.

The gradient $\nabla\nu(x)$ is continuous since ν is continuously differentiable (as μ_A is continuously differentiable).

The persistence $P(x) = 1/(1 + |\nabla\nu(x)|)$ is continuous since the denominator is always positive ($1 + |\nabla\nu(x)| \geq 1 > 0$).

The transience $T(x) = |\nabla\nu(x)|/(1 + |\nabla\nu(x)|)$ is continuous.

Now, each probability is a product of continuous functions:

$$\begin{aligned} p_M(x) &= |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)>0\}}, & (64)\\ p_P(x) &= \nu(x)P(x), & (65)\\ p_T(x) &= \nu(x)T(x), & (66)\\ p_N(x) &= |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)<0\}}. & (67)\end{aligned}$$

The indicator functions $\mathbf{1}_{\{\delta(x)>0\}}$ and $\mathbf{1}_{\{\delta(x)<0\}}$ are discontinuous at $\delta(x) = 0$. However, at $\delta(x) = 0$, the factor $|\delta(x)| = 0$, so the products $p_M(x)$ and $p_N(x)$ are continuous through the zero.

Thus all four probabilities are continuous functions of μ_A. □

Remark 3.15. The intrinsic probabilities are entirely determined by the membership function and its local geometry. No neighborhoods, similarity measures, or external evidence mechanisms are required. The probabilities emerge directly from the uncertainty structure of each element.

3.4 Interpretation of the Probability Quadruple

The quadruple $\Pi(x) = (p_M(x), p_P(x), p_T(x), p_N(x))$ admits a clear semantic interpretation:

- $p_M(x)$: The intrinsic probability that x belongs to the membership region. This is high when commitment favors membership and low when the element is near equilibrium.
- $p_N(x)$: The intrinsic probability that x belongs to the non-membership region. This is high when commitment favors non-membership and low when the element is near equilibrium.

- $p_P(x)$: The intrinsic probability that x exhibits persistent equilibrium. This is high when fuzziness is high and locally constant.
- $p_T(x)$: The intrinsic probability that x exhibits transient equilibrium. This is high when fuzziness is high and locally varying.

The sum of all four probabilities is identically 1 for every $x \in U$.

3.5 *Individualized Uncertainty Profiles*

A distinguishing feature of the intrinsic probabilistic representation is that each element possesses its own uncertainty profile:

$$\Pi(x) = (p_M(x), p_P(x), p_T(x), p_N(x)) \,. \tag{68}$$

This profile is entirely determined by the local geometry of the membership function at x. Two elements with identical membership values $\mu_A(x_1) = \mu_A(x_2)$ may have different probability quadruples if their fuzziness gradients differ.

Theorem 3.16 (Individualization). *The probability quadruple $\Pi(x)$ is not determined by $\mu_A(x)$ alone, but by the local geometry of μ_A as captured by $\nabla\nu(x)$.*

Proof. The probabilities depend on $\nu(x)$ and $\nabla\nu(x)$. While $\nu(x)$ is determined by $\mu_A(x)$, $\nabla\nu(x)$ depends on the behavior of μ_A in the neighborhood of x. Therefore two elements with identical $\mu_A(x)$ but different gradients will have different probability quadruples. □

Remark 3.17. To illustrate, consider the continuous piecewise linear membership function

$$\mu_A(x) = \begin{cases} 0.7 + 0.8(x - 0.5), & 0 \le x \le 0.5, \\ 0.7 - 1.2(x - 0.5), & 0.5 \le x \le 1. \end{cases}$$

The equation $\mu_A(x) = 0.6$ yields two solutions:

$$x_1 = 0.5 - \frac{0.1}{0.8} = 0.375, \qquad x_2 = 0.5 + \frac{0.1}{1.2} \approx 0.5833.$$

Thus $\mu_A(x_1) = \mu_A(x_2) = 0.6$. However, $\mu'_A(x_1) = 0.8$ while $\mu'_A(x_2) = -1.2$. Consequently, $|\nabla\nu(x_1)| = 2|\mu'_A(x_1)| = 1.6$ and $|\nabla\nu(x_2)| = 2|\mu'_A(x_2)| = 2.4$. Hence $P(x_1) = 0.3846$ whereas $P(x_2) = 0.2941$. Since $\nu(x_1) = \nu(x_2) = 0.8$, the equilibrium probabilities differ:

$$\big(p_P(x_1), p_T(x_1)\big) = (0.3077, 0.4923), \qquad \big(p_P(x_2), p_T(x_2)\big) = (0.2353, 0.5647).$$

Both points share the same commitment probability $p_C = 0.2$, yet their full quadruples differ:

$$\Pi(x_1) = (0.2,\ 0.3077,\ 0.4923,\ 0), \qquad \Pi(x_2) = (0.2,\ 0.2353,\ 0.5647,\ 0).$$

Thus $\Pi(x_1) \neq \Pi(x_2)$ despite identical membership values. The difference arises entirely from the distinct local geometries at the two points.

Summary of Section

In this section, we have:

1. Observe that uncertainty is a compositional structure with three fundamental components.
2. Discuss the intrinsic evidence components: $|\delta(x)|$, $\nu(x)P(x)$, and $\nu(x)T(x)$.
3. Establish that these components form a partition of unity.
4. Defined intrinsic probabilities directly from evidence components.

5. Refined the three-state distribution to a four-state distribution.

6. Proved conservation, equilibrium maximality, commitment dominance, persistence dominance, transience dominance, and continuity.

7. Established individualization as a key property.

All quantities are derived from the membership function and its geometry. No neighborhoods are required. No external parameters are introduced. The probabilities emerge directly from the uncertainty structure of each element.

In the next section, we will integrate these intrinsic probabilities into the framework of Probabilistic Shadowed Sets.

4. Probabilistic Shadowed Sets

4.1 Observational Foundation

In Section 2, we established that uncertainty possesses an intrinsic structural geometry, yielding four structural states: membership (M), persistent equilibrium ($\mathcal{E}_P$), transient equilibrium ($\mathcal{E}_T$), and non-membership (N). In Section 3, we derived intrinsic probabilities $\Pi(x) = (p_M(x), p_P(x), p_T(x), p_N(x))$ that emerge directly from the uncertainty structure of each individual element.

We now integrate these developments into the framework of shadowed sets. The fundamental observation that motivates this integration is:

A shadowed set is an approximation of a fuzzy set that preserves its essential uncertainty structure while reducing representational complexity.

Classical shadowed sets achieve this by partitioning the universe into three regions: acceptance (core), rejection (exclusion), and shadow (uncertainty). However, as we have observed, the shadow region is structurally heterogeneous. The distinction between persistent and transient equilibrium reveals that uncertainty is not a single phenomenon.

The Probabilistic Shadowed Set (PSS) extends the classical framework by: (i) preserving the structural distinction within the shadow region; (ii) maintaining the fundamental principle of uncertainty retention; and (iii) deriving all quantities from first principles.

4.2 Definition of Probabilistic Shadowed Sets

The intrinsic probabilistic representation developed in the preceding section induces a natural extension of the classical shadowed set framework.

Definition 4.1 (Probabilistic Shadowed Set)**.** Let A be a fuzzy set on U with membership function $\mu_A : U \to [0, 1]$.

For each $x \in U$, define the equilibrium deviation $\delta(x) = 2\mu_A(x) - 1$ and the fuzziness degree $\nu(x) = 1 - |\delta(x)|$. The quantities $|\delta(x)|$ and $\nu(x)$ represent the committed informational mass and uncertainty mass, respectively, with $|\delta(x)| + \nu(x) = 1$.

To characterize the internal organization of uncertainty, define the persistence coefficient $P(x) = 1/(1 + |\nabla\nu(x)|)$ and the transience coefficient $T(x) = 1 - P(x)$. These partition the uncertainty mass: $\nu(x) = \nu(x)P(x) + \nu(x)T(x)$.

The four structural probability components are:

$$p_M(x) = \begin{cases} |\delta(x)|, & \delta(x) > 0, \\ 0, & \delta(x) \le 0, \end{cases} \qquad p_N(x) = \begin{cases} |\delta(x)|, & \delta(x) < 0, \\ 0, & \delta(x) \ge 0, \end{cases}$$

$$p_P(x) = \nu(x)P(x), \qquad p_T(x) = \nu(x)T(x).$$

These satisfy $p_M(x) + p_P(x) + p_T(x) + p_N(x) = 1$ and all components are non-negative.
Consequently,

$$\Pi_A(x) = \big(p_M(x),\, p_P(x),\, p_T(x),\, p_N(x)\big)$$

defines a probability distribution on $\Omega_x = \{M, P, T, N\}$, where M denotes membership, P persistent uncertainty, T transient uncertainty, and N non-membership.

The mapping

$$\Pi_A : U \to \Delta^3,$$

where Δ^3 denotes the probability simplex in $\mathbb{R}^4$, is called the *Probabilistic Shadowed Set* induced by A.

In contrast to classical shadowed sets, which collapse all uncertainty into a single homogeneous shadow region, the Probabilistic Shadowed Set assigns to each element a probability distribution over four mutually exclusive structural states. Thus uncertainty is represented by both its magnitude and its structural organization across commitment, persistent equilibrium, and transient equilibrium.

Remark 4.2. The Probabilistic Shadowed Set generalizes the classical shadowed set in two fundamental ways: the codomain is $[0, 1]^4$ rather than $\{0, a, 1\}$, providing graded structural affiliations; and the shadow region is decomposed into persistent and transient components, preserving structural information that is lost in the classical formulation.

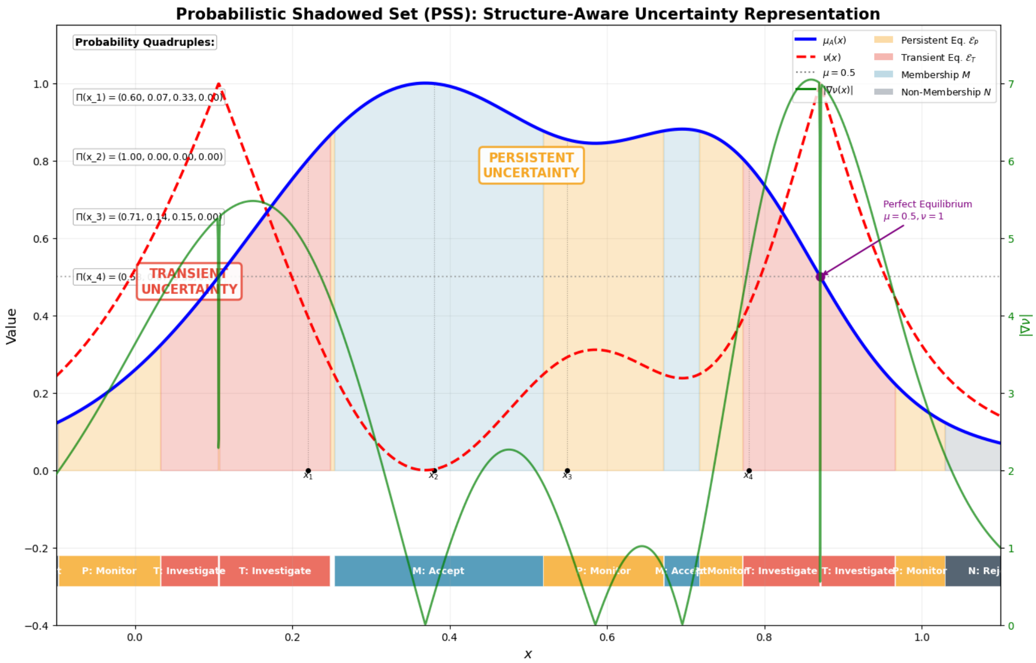

Fig. 2. Probabilistic Shadowed Set (PSS) schema illustrating the four structural states induced by the geometry of a smooth fuzzy membership function. The blue solid curve is the membership function $\mu_A(x)$, the red dashed curve is the fuzziness degree $\nu(x) = 1 - |2\mu_A(x) - 1|$, and the gray dotted line marks the equilibrium $\mu = 0.5$ (where $\nu = 1$). The green solid curve (right axis) shows the gradient magnitude $|\nabla\nu(x)|$, capturing the local variation of fuzziness. Coloured shaded regions under μ_A denote: M (blue) – membership, P (orange) – persistent equilibrium, T (red) – transient equilibrium, and N (grey) – non-membership. At four sample points $x_1, \ldots, x_4$, the probability quadruples $\Pi(x_i) = \big(p_M(x_i), p_P(x_i), p_T(x_i), p_N(x_i)\big)$ are displayed, quantifying the intrinsic probabilities of each state. The annotations "PERSISTENT UNCERTAINTY" and "TRANSIENT UNCERTAINTY" highlight the nature of the equilibrium zones. The bottom colour bar maps each region to the corresponding decision action: **Accept** (M), **Monitor** (P), **Investigate** (T), and **Reject** (N).

Figure 2 illustrates the complete PSS decomposition for a representative smooth membership function, showing the four structural states and the corresponding probability quadruples at selected points.

4.3 *Structural Non-Invertibility*

A fundamental property of the Probabilistic Shadowed Set is that the mapping from the probabilistic representation to the membership value is not invertible: knowing $\mu_A(x)$ alone is insufficient to recover

$\Pi_A(x)$.

Definition 4.3 (Projection Mapping)**.** The projection $F : \Pi_A(x) \mapsto \mu_A(x)$ extracts the membership value from the probabilistic representation.

Theorem 4.4 (Structural Non-Invertibility)**.** *The projection F is surjective but not injective:*

1. *Every $\mu_A(x)$ has at least one $\Pi_A(x)$.*
2. *There exist $\Pi_A(x_1) \neq \Pi_A(x_2)$ such that $F(\Pi_A(x_1)) = F(\Pi_A(x_2))$.*

Proof. Surjectivity follows from the construction: any $\mu_A(x)$ yields a corresponding $\Pi_A(x)$. Non-injectivity follows because $P(x)$ depends on $\nabla\nu(x)$, which is not determined by $\mu_A(x)$ alone. Two points with identical $\mu_A(x)$ but different gradients yield different persistence values and hence different probability quadruples (see Remark 3.17 for an explicit illustration). □

Remark 4.5. Structural non-invertibility demonstrates that PSS contains information not recoverable from $\mu_A(x)$ alone. The PSS is therefore a strict refinement of conventional fuzzy membership, preserving structural information lost when only the membership value is considered.

4.4 *Balance Condition and Threshold Determination*

The fundamental principle underlying shadowed sets is the preservation of total uncertainty. In classical shadowed sets, this is expressed through Pedrycz's balance condition:

$$\phi(M) + \phi(N) = \phi(\mathcal{E}),$$

where $\phi(S)$ denotes the fuzziness contributed by region S.

In the Probabilistic Shadowed Set framework, a natural analogue is the requirement that the total probability mass assigned to committed states equals the total probability mass assigned to equilibrium states:

$$\sum_{x\in U}\big(p_M(x) + p_N(x)\big) = \sum_{x\in U}\big(p_P(x) + p_T(x)\big). \tag{69}$$

Using the definitions of the structural probabilities, this condition simplifies to:

$$\sum_{x}\big(p_M(x) + p_N(x)\big) = \sum_{x}|\delta(x)|, \tag{70}$$

$$\sum_{x}\big(p_P(x) + p_T(x)\big) = \sum_{x}\nu(x). \tag{71}$$

Thus the balance condition becomes:

$$\sum_{x\in U}|\delta(x)| = \sum_{x\in U}\nu(x). \tag{72}$$

This expresses that the total commitment probability equals the total uncertainty probability, which is a natural and interpretable balance condition for the PSS framework.

4.4.1 *Expected Fuzziness*

To quantify the average uncertainty remaining after the probabilistic decomposition, we define the expected fuzziness of the Probabilistic Shadowed Set as

$$\phi(\Pi_A) = \sum_{x\in U}\sum_{S\in\{M,P,T,N\}} p_S(x)\cdot\nu_S(x), \tag{73}$$

where $\nu_S(x)$ is the fuzziness degree associated with state S: $\nu_M(x) = \nu_N(x) = 0$ for committed states, and $\nu_P(x) = \nu_T(x) = \nu(x)$ for equilibrium states.

Theorem 4.6 (Expected Fuzziness). *For any Probabilistic Shadowed Set Π_A defined by the intrinsic probabilities,*

$$\phi(\Pi_A) = \sum_{x \in U} \nu(x)^2. \tag{74}$$

Proof. The expected fuzziness of the PSS is:

$$\phi(\Pi_A) = \sum_{x \in U} (p_P(x)\nu(x) + p_T(x)\nu(x)) \tag{75}$$

$$= \sum_{x \in U} (p_P(x) + p_T(x))\, \nu(x) \tag{76}$$

$$= \sum_{x \in U} \nu(x)\big(P(x) + T(x)\big)\nu(x) \tag{77}$$

$$= \sum_{x \in U} \nu(x)^2. \tag{78}$$

The penultimate equality follows from $p_P(x) = \nu(x)P(x)$ and $p_T(x) = \nu(x)T(x)$, and the final equality from $P(x) + T(x) = 1$. □

Remark 4.7. The expected fuzziness $\phi(\Pi_A) = \sum_x \nu(x)^2$ differs from the total fuzziness of the original fuzzy set $\phi(A) = \sum_x \nu(x)$. This is not an inconsistency but a consequence of the probabilistic interpretation: $\phi(\Pi_A)$ is the expected value of the fuzziness under the probability distribution over structural states, not the sum of the actual fuzziness values.

4.4.2 Threshold Determination

The structural thresholds η and τ define the partition of U into the four states. Recall the definitions:

$$M = \{x : \nu(x) < \eta,\ \delta(x) > 0\}, \quad N = \{x : \nu(x) < \eta,\ \delta(x) < 0\},$$
$$\mathcal{E}_P = \{x : \nu(x) \geq \eta,\ P(x) \geq \tau\}, \quad \mathcal{E}_T = \{x : \nu(x) \geq \eta,\ P(x) < \tau\}.$$

The threshold η separates committed states (M, N) from equilibrium states ($\mathcal{E}_P, \mathcal{E}_T$). The threshold τ then separates the equilibrium region into persistent and transient components.

Critically, τ does not affect the balance between committed and equilibrium states — it only determines how the equilibrium probability mass is distributed between P and T. Therefore, the balance condition determines η independently of τ.

Remark 4.8 (Threshold Separation). The balance condition $\sum_x |\delta(x)| = \sum_x \nu(x)$ determines the threshold η independently of τ. From the definitions:

$$\sum_x \big(p_M(x) + p_N(x)\big) = \sum_{x:\nu(x)<\eta} |\delta(x)|, \qquad \sum_x \big(p_P(x) + p_T(x)\big) = \sum_{x:\nu(x)\geq\eta} \nu(x).$$

Neither expression depends on τ. Thus the balance condition involves only η. Once η is determined, the equilibrium region $\mathcal{E} = \{x : \nu(x) \geq \eta\}$ is fixed, and τ can be chosen to achieve any desired split between persistent and transient equilibrium.

This separation has two important consequences. First, the balance condition reduces to a one-dimensional search over η, which is computationally efficient and analogous to the single-threshold optimization in classical shadowed sets. Second, τ provides additional flexibility: it allows the user to control the relative emphasis on persistent versus transient uncertainty without affecting the fundamental balance between commitment and equilibrium.

In the default setting, a natural choice is

$$\tau = \mathrm{median}\{P(x) : x \in \mathcal{E}\},$$

which partitions the equilibrium region into two components of equal persistence. This yields a balanced decomposition of the shadow region without requiring additional parameters.

Definition 4.9 (Balance Objective). For a given threshold η, the balance objective is defined as

$$\mathcal{B}(\eta) = \left| \sum_{x:\nu(x)<\eta} |\delta(x)| - \sum_{x:\nu(x)\geq\eta} \nu(x) \right|. \tag{79}$$

The optimal threshold is then chosen as $\eta^* = \arg\min_\eta \mathcal{B}(\eta)$.

Theorem 4.10 (Existence of Optimal Threshold). *For any fuzzy set A defined on a finite universe U, there exists a threshold η^* that minimizes $\mathcal{B}(\eta)$.*

Proof. The universe U is finite, so there are only finitely many distinct values of $\nu(x)$. Therefore, there are only finitely many threshold values η that yield distinct partitions of U into committed and equilibrium regions. Since $\mathcal{B}(\eta)$ takes only finitely many values, a minimum exists. □

The threshold determination procedure is therefore:

1. Determine η^* by minimizing $\mathcal{B}(\eta)$.
2. Define the equilibrium region $\mathcal{E} = \{x : \nu(x) \geq \eta^*\}$.
3. Choose τ (default: median persistence in $\mathcal{E}$) to split $\mathcal{E}$ into $\mathcal{E}_P$ and $\mathcal{E}_T$.

Remark 4.11. The separation of thresholds is a key feature of the framework. The balance between commitment and equilibrium is determined solely by η. The threshold τ controls the internal structure of the equilibrium region without affecting the fundamental balance. This two-stage procedure ensures computational efficiency while preserving the descriptive richness of the four-state representation. Alternative choices of τ may be specified by the user to reflect domain-specific requirements.

4.5 Shadow Aggregation and Reduction

Definition 4.12 (Shadow Aggregation). Let Π_A be a Probabilistic Shadowed Set. The aggregated shadow probability is defined as

$$p_{\mathcal{E}}(x) = p_P(x) + p_T(x) = \nu(x). \tag{80}$$

Thus, the total probability assigned to equilibrium coincides exactly with the fuzziness degree.

Theorem 4.13 (Reduction to a Three-State Representation). *Let $\Pi_A(x) = (p_M(x), p_P(x), p_T(x), p_N(x))$ be a Probabilistic Shadowed Set. If persistent and transient equilibrium are merged into a single equilibrium state, then $\Pi_A(x)$ reduces to*

$$\Pi_A^{(3)}(x) = (p_M(x), p_{\mathcal{E}}(x), p_N(x)),$$

where $p_{\mathcal{E}}(x) = p_P(x) + p_T(x) = \nu(x)$. Furthermore, $p_M(x) + p_{\mathcal{E}}(x) + p_N(x) = 1$ for every $x \in U$.

Proof. By definition, $p_P(x) = \nu(x)P(x)$ and $p_T(x) = \nu(x)T(x)$. Since $P(x) + T(x) = 1$, it follows that $p_{\mathcal{E}}(x) = \nu(x)$. Thus $\Pi_A^{(3)}(x) = (p_M(x), \nu(x), p_N(x))$. Finally, $p_M(x) + p_N(x) = |\delta(x)|$ and $\nu(x) = 1 - |\delta(x)|$, so $p_M(x) + p_{\mathcal{E}}(x) + p_N(x) = |\delta(x)| + \nu(x) = 1$. □

Remark 4.14. The reduction theorem establishes that Probabilistic Shadowed Sets contain the classical shadow interpretation as a coarser representation obtained by merging persistent and transient equilibrium. However, the threshold of a classical shadowed set is not determined by a universal closed-form expression. It must be obtained through the chosen shadowed-set optimization principle, such as uncertainty balance, minimum error, or related criteria. In this sense, PSS is a strict refinement of CSS: it preserves the three-state structure while adding the ability to distinguish persistent from transient uncertainty.

4.6 Information Decomposition and Preservation

A fundamental question is whether the refinement introduced by the Probabilistic Shadowed Set representation genuinely preserves additional information relative to its aggregated counterpart. The answer is affirmative.

Recall that the aggregated shadow probability is $p_{\mathcal{E}}(x) = p_P(x) + p_T(x) = \nu(x)$. The corresponding aggregated representation is

$$\Pi_A^{\text{agg}}(x) = \big(p_M(x), p_{\mathcal{E}}(x), p_N(x)\big),$$

which ignores the distinction between persistent and transient equilibrium.

The following result establishes that the full Probabilistic Shadowed Set representation always contains at least as much information as its aggregated reduction.

Theorem 4.15 (Information Preservation)**.** *Let $\Pi_A(x) = (p_M(x), p_P(x), p_T(x), p_N(x))$ be the Probabilistic Shadowed Set, and let $\Pi_A^{\text{agg}}(x) = (p_M(x), p_{\mathcal{E}}(x), p_N(x))$ be its aggregated reduction, where $p_{\mathcal{E}}(x) = p_P(x) + p_T(x)$. Then $H(\Pi_A(x)) \geq H(\Pi_A^{\text{agg}}(x))$ for all $x \in U$. Moreover,*

$$H(\Pi_A(x)) - H(\Pi_A^{\text{agg}}(x)) = p_{\mathcal{E}}(x) H\left(\frac{p_P(x)}{p_{\mathcal{E}}(x)}, \frac{p_T(x)}{p_{\mathcal{E}}(x)}\right).$$

Proof. By the grouping property of Shannon entropy applied to $p_{\mathcal{E}}(x) = p_P(x) + p_T(x)$,

$$H(\Pi_A(x)) = H(\Pi_A^{\text{agg}}(x)) + p_{\mathcal{E}}(x) H\left(\frac{p_P(x)}{p_{\mathcal{E}}(x)}, \frac{p_T(x)}{p_{\mathcal{E}}(x)}\right).$$

Since entropy is nonnegative, the second term is ≥ 0, hence $H(\Pi_A(x)) \geq H(\Pi_A^{\text{agg}}(x))$. □

The theorem provides an exact decomposition of the informational content of a Probabilistic Shadowed Set. The quantity

$$\Delta H(x) = p_{\mathcal{E}}(x) H\left(\frac{p_P(x)}{p_{\mathcal{E}}(x)}, \frac{p_T(x)}{p_{\mathcal{E}}(x)}\right)$$

measures the uncertainty associated with the internal structure of the equilibrium region. Consequently, the distinction between persistent and transient equilibrium is not merely a semantic refinement but corresponds to a quantifiable amount of information that is absent from the aggregated representation.

From this perspective, the probabilistic decomposition reveals a layer of structural uncertainty that is invisible to classical shadowed sets. The additional information preserved by the Probabilistic Shadowed Set is precisely the information required to characterize how equilibrium is distributed between sustained and transitional forms.

Corollary 4.16 (Additional Information Bound)**.** *For every $x \in U$,*

$$0 \leq \Delta H(x) \leq p_{\mathcal{E}}(x) \leq 1,$$

where $\Delta H(x) = H(\Pi_A(x)) - H(\Pi_A^{\text{agg}}(x))$.

Proof. From Theorem 4.15,

$$\Delta H(x) = p_{\mathcal{E}}(x) H\left(\frac{p_P(x)}{p_{\mathcal{E}}(x)}, \frac{p_T(x)}{p_{\mathcal{E}}(x)}\right).$$

The entropy of a binary distribution is at most 1, so $\Delta H(x) \leq p_{\mathcal{E}}(x)$. Since $p_{\mathcal{E}}(x) = \nu(x) \leq 1$, the result follows. □

Remark 4.17. The information gain induced by the persistent–transient decomposition is

$$\Delta H(x) = \nu(x) H(P(x), T(x)).$$

This expression reveals that the additional information is governed jointly by the degree of uncertainty $\nu(x)$ and the structural allocation of that uncertainty between persistence and transience. The gain vanishes whenever $\nu(x) = 0$ or when all equilibrium mass is concentrated in a single component, and it is maximized when uncertainty is distributed equally between persistent and transient equilibrium.

4.7 PSS Decision Policies

The probabilistic decomposition assigns to each element $x \in U$ a probability quadruple

$$\Pi_A(x) = \big(p_M(x), p_P(x), p_T(x), p_N(x)\big),$$

where $p_M(x)$ and $p_N(x)$ quantify commitment toward membership and non-membership, while $p_P(x)$ and $p_T(x)$ quantify persistent and transient equilibrium, respectively.

A natural question arises: how should decisions be made from this representation? In classical shadowed sets, decision making is achieved through a three-way partition of the universe into membership, shadow, and non-membership regions. The shadow region represents a deferred decision. In contrast, the Probabilistic Shadowed Set reveals that not all deferred decisions are equivalent. Some correspond to stable equilibrium, whereas others correspond to transitional equilibrium. Consequently, the probabilistic decomposition naturally induces a four-way decision model.

4.7.1 Dominant-State Policy

The most direct decision policy assigns each element to the state with highest probability.

Definition 4.18 (Dominant-State Decision)**.** Given a Probabilistic Shadowed Set representation $\Pi_A(x)$, the dominant-state decision is defined by

$$D_{\max}(x) = \arg\max\{p_M(x), p_P(x), p_T(x), p_N(x)\}.$$

This rule assigns every element to exactly one of the four structural states: M, P, T, N. Unlike threshold-based constructions, the decision emerges directly from the intrinsic probabilistic structure of the fuzzy set.

4.7.2 Decision Semantics

The four probabilities admit a natural interpretation in terms of actions:

Table 1
Decision semantics of Probabilistic Shadowed Sets

State	Recommended Action	Nature of Action
M	Accept membership	Commitment
P	Monitor	Passive observation
T	Investigate	Active information gathering
N	Reject membership	Commitment

The distinction between P and T is fundamental. Persistent equilibrium arises when uncertainty remains stable across a region of the universe. Such uncertainty is structural and may not be resolved easily through additional information. The appropriate action is to defer and monitor passively. Transient equilibrium, on the other hand, occurs near rapid changes of fuzziness and signals an ongoing transition. The appropriate action is to investigate actively or acquire evidence.

4.7.3 Confidence-Based Policy

In certain applications, a decision may be made only when sufficient confidence exists.

Definition 4.19 (Confidence-Based Decision)**.** Let $\lambda \in [0, 1]$ be a confidence threshold. The decision policy is

$$D_\lambda(x) = \begin{cases} M, & p_M(x) \geq \lambda, \\ P, & p_P(x) \geq \lambda, \\ T, & p_T(x) \geq \lambda, \\ N, & p_N(x) \geq \lambda, \\ \text{undetermined}, & \text{otherwise}. \end{cases}$$

This policy introduces a fifth outcome corresponding to insufficient confidence, which may be useful in safety-critical applications where premature commitment is undesirable.

4.7.4 Expected Utility Policy

When application-specific utilities are available, decisions may be selected according to expected utility. Let $u_{a,S}$ denote the utility of taking action $a \in \{M, P, T, N\}$ when the true state is $S \in \{M, P, T, N\}$. The expected utility of action a at x is:

$$U_a(x) = \sum_{S \in \{M,P,T,N\}} p_S(x) \cdot u_{a,S}. \tag{81}$$

The optimal decision is then:

$$D_U(x) = \arg \max_{a \in \{M,P,T,N\}} U_a(x). \tag{82}$$

Remark 4.20. The expected utility policy is the most flexible decision rule. It enables the incorporation of application-specific preferences while preserving the intrinsic probabilistic structure of the Probabilistic Shadowed Set.

4.8 Algorithm for Probabilistic Shadowed Set Construction

We present the complete algorithm for constructing a Probabilistic Shadowed Set from a given fuzzy set.

Table 2
Construction of the Probabilistic Shadowed Set

Input: Fuzzy set $A = \{(x, \mu_A(x)) : x \in U\}$
Output: $\Pi_A(x) = \big(p_M(x), p_P(x), p_T(x), p_N(x)\big)$

Step 1. Compute equilibrium deviation
For each $x \in U$
$\delta(x) = 2\mu_A(x) - 1$
$\nu(x) = 1 - |\delta(x)|$

Step 2. Compute persistence and transience
For each $x \in U$
Compute $\nabla\nu(x)$ using finite differences
$P(x) = \dfrac{1}{1 + |\nabla\nu(x)|}$
$T(x) = 1 - P(x)$

Step 3. Compute probabilistic components
For each $x \in U$
$p_M(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)>0\}}$
$p_N(x) = |\delta(x)| \cdot \mathbf{1}_{\{\delta(x)<0\}}$
$p_P(x) = \nu(x)P(x)$
$p_T(x) = \nu(x)T(x)$

Return $\Pi_A(x) = \big(p_M(x), p_P(x), p_T(x), p_N(x)\big)$

Remark 4.21. The algorithm is entirely data-driven. All quantities are derived directly from the membership function $\mu_A(x)$ and its local geometry. No external parameters, thresholds, or optimization procedures are required. The conservation property $\sum_S p_S(x) = 1$ follows directly from the definitions.

4.9 Global Partition Thresholds

A natural question is whether the PSS framework admits global thresholds $\alpha, \beta_P, \beta_T, \gamma$ that partition U into $M, \mathcal{E}_P, \mathcal{E}_T, N$ using only $\mu_A(x)$. The answer is negative.

The PSS representation depends not only on $\mu_A(x)$ but also on the local geometry through $P(x) = 1/(1 + |\nabla\nu(x)|)$. Two elements with identical membership values may have different persistence values and hence different probability quadruples.

Theorem 4.22 (Nonexistence of Universal Four-Way Thresholds). *No global thresholds $\alpha, \beta_P, \beta_T, \gamma$ exist such that the four PSS regions can be recovered solely from inequalities involving $\mu_A(x)$.*

Proof. If such thresholds existed, classification would depend only on $\mu_A(x)$. But by the Structural Non-Invertibility Theorem, there exist x_1, x_2 with $\mu_A(x_1) = \mu_A(x_2)$ yet $\Pi_A(x_1) \neq \Pi_A(x_2)$. Thus no μ_A-based rule can reproduce the PSS decomposition. □

The absence of universal thresholds reflects that PSS captures information unavailable to membership-based representations.

A related question concerns a three-way approximation obtained by merging P and T into a single shadow state. Since $p_{\mathcal{E}}(x) = p_P(x) + p_T(x) = \nu(x)$, the reduced representation is $\Pi_A^{(3)}(x) = (p_M(x), \nu(x), p_N(x))$. An exact global threshold generally does not exist because $\nu(x)$ is continuously varying.

Nevertheless, one may approximate the average uncertainty by choosing α to match the classical shadow proportion:

$$\alpha^* = \arg \min_{\alpha \in [0,1/2]} \left| \frac{1}{|U|} \sum_{x \in U} \mathbf{1}_{\{\alpha \leq \mu_A(x) \leq 1-\alpha\}} - \frac{1}{|U|} \sum_{x \in U} \nu(x) \right| .$$

No universal closed-form expression for α^* exists; it must be computed numerically, except for special cases such as symmetric functions with compact support.

Thus, unlike classical shadowed sets, PSS does not admit universal global partition thresholds. Its decomposition is inherently element-wise, arising from local uncertainty geometry rather than fixed membership cut levels.

Summary

The Probabilistic Shadowed Set framework generalizes classical shadowed sets in three respects: (i) all quantities emerge from the geometry of the membership function; (ii) the shadow region is decomposed into persistent and transient components, preserving structural information; and (iii) each element possesses its own uncertainty profile, determined by its local geometry. The framework thus provides a principled, first-principles representation of uncertainty that respects the intrinsic geometry of fuzzy membership.

5. Validation of the Probabilistic Shadowed Set Framework

The objective of this section is to examine whether the fundamental principles underlying Probabilistic Shadowed Sets (PSS) are manifested in practice. The validation is organized around four questions: (i) Does the persistent-transient decomposition yield a richer informational representation than classical shadowed sets? (ii) Is the framework robust to the choice of persistence operator? (iii) Is the decomposition stable under perturbations? (iv) Is the framework computationally feasible?

All experiments were conducted on the domain $x \in [-4, 4]$ using continuous membership functions with diverse geometric characteristics: Gaussian (smooth, unimodal, exponentially decaying), triangular (piecewise linear, compact support), trapezoidal (piecewise linear with plateau), and sigmoidal (smooth monotonic transition). These functions span a representative range of fuzzy set geometries commonly encountered in practice.

Figure 3 provides a compact overview of the benchmark membership geometries together with their persistence profiles. The comparison emphasizes that persistence is a geometric property: similar membership magnitudes can be associated with different local stability patterns when the surrounding shape of the fuzzy set changes.

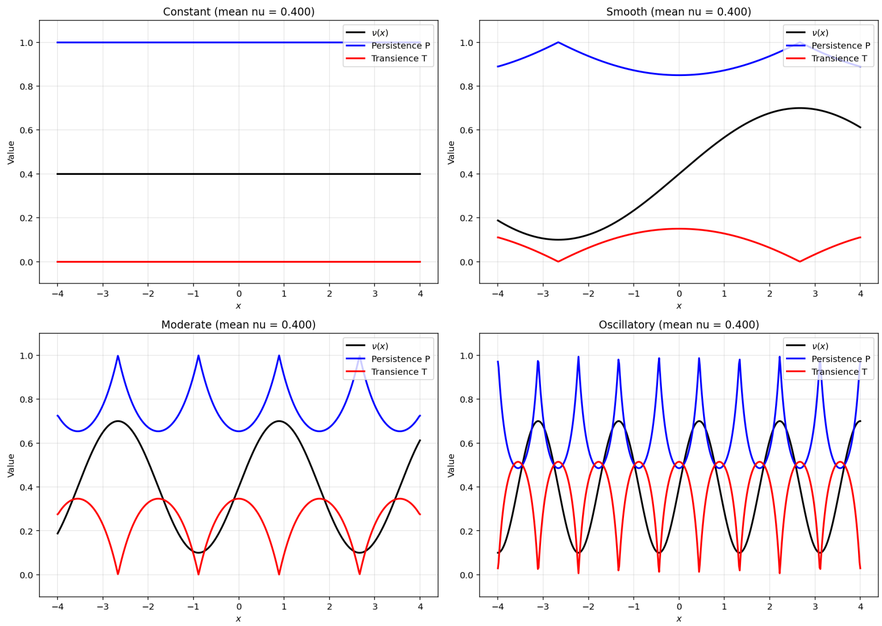

Fig. 3. Membership functions and corresponding persistence profiles for Gaussian, triangular, trapezoidal, and sigmoidal functions. Persistence is highest where fuzziness is locally stable

This overview motivates the separate validation of probability conservation, information gain, operator sensitivity, perturbation robustness, and computational cost in the subsections that follow.

5.1 *Theoretical Consistency Validation*

To validate the intrinsic probabilistic structure derived in this section, we consider a standard Gaussian fuzzy set defined on the domain $x \in [-4, 4]$:

$$\mu_A(x) = \exp\left(-\frac{x^2}{2}\right). \tag{83}$$

For this fuzzy set, we compute all intrinsic quantities derived above. Figure 4 presents the results in four panels.

The numerical validation confirms the following identities to machine precision:

Table 3
Validation of intrinsic probability identities

Identity	Max Error	Status
$p_C + \nu = 1$	0.00×10^{0}	Exact
$P + T = 1$	0.00×10^{0}	Exact
$p_M + p_N = p_C$	0.00×10^{0}	Exact
$p_C + p_P + p_T = 1$	1.11×10^{-16}	Exact
$p_M + p_P + p_T + p_N = 1$	1.11×10^{-16}	Exact
$p_P + p_T = \nu$	1.11×10^{-16}	Exact

At the equilibrium point $\mu_A(x) \approx 0.5$ (the center of the Gaussian), the computed quantities are:

$$\mu_A \approx 0.4978, \tag{84}$$
$$\delta \approx -0.0044, \tag{85}$$
$$\nu \approx 0.9956, \tag{86}$$
$$p_C \approx 0.0044, \tag{87}$$
$$p_P \approx 0.6412, \tag{88}$$
$$p_T \approx 0.3543, \tag{89}$$
$$p_P + p_T \approx 0.9956. \tag{90}$$

These results reveal several important insights. First, at equilibrium, commitment probability is nearly zero and fuzziness is near maximum. Second, the fuzziness is split between persistent equilibrium (64%) and transient equilibrium (36%), reflecting the local geometry of the Gaussian membership function. Third, the fundamental identity $p_P + p_T = \nu$ holds exactly.

We also tested a subnormal Gaussian fuzzy set with height $h = 0.8$:

$$\mu_A(x) = 0.8 \exp\left(-\frac{x^2}{2}\right). \tag{91}$$

At the maximum fuzziness point $\mu_A \approx 0.4$, the framework yields:

$$\mu_A \approx 0.3982, \tag{92}$$
$$\delta \approx -0.2035, \tag{93}$$
$$\nu \approx 0.7965, \tag{94}$$
$$p_C \approx 0.2035, \tag{95}$$
$$p_P \approx 0.4104, \tag{96}$$
$$p_T \approx 0.3861, \tag{97}$$
$$p_P + p_T \approx 0.7965. \tag{98}$$

This confirms that the framework handles subnormal fuzzy sets naturally. The maximum fuzziness point need not coincide with $\mu_A = 0.5$; the framework adapts to the geometry of the membership function.

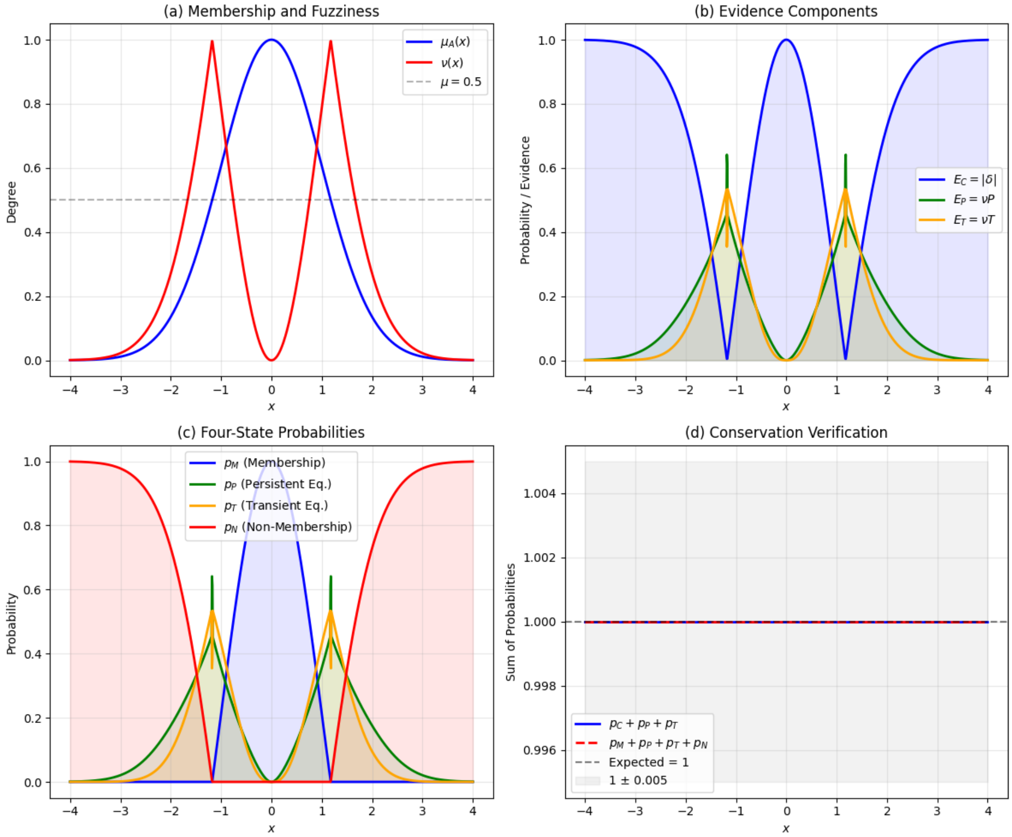

Fig. 4. Validation of the intrinsic probabilistic structure on a Gaussian fuzzy set. (a) Membership function $\mu_A(x)$ and fuzziness degree $\nu(x)$. (b) The three evidence components: commitment evidence $E_C = |\delta|$, persistent equilibrium evidence $E_P = \nu P$, and transient equilibrium evidence $E_T = \nu T$. These components partition unity: $E_C + E_P + E_T = 1$. (c) The four-state probabilities: membership p_M, persistent equilibrium p_P, transient equilibrium p_T, and non-membership p_N. (d) Conservation verification: $p_C + p_P + p_T = 1$ and $p_M + p_P + p_T + p_N = 1$

The validation demonstrates that the probabilistic representation is not an approximation but an exact consequence of the geometry of the membership function. The conservation laws hold to machine precision, confirming the mathematical rigor of the framework.

These numerical results confirm that the intrinsic probabilities form a complete probability distribution, establishing the Probabilistic Shadowed Set representation:

$$\Pi(x) = (p_M(x), p_P(x), p_T(x), p_N(x)), \tag{99}$$

where the four components are as defined in the preceding subsection. This representation is now ready for application in decision-making and threshold induction.

5.2 *Comparison with Classical Shadowed Sets*

The primary distinction between PSS and existing shadowed-set models lies in the treatment of equilibrium. In Pedrycz's uncertainty-balance shadowed sets and Gao's Mean Entropy Shadowed Sets (MESS), the equilibrium region is represented by a single component E. In contrast, PSS decomposes equilibrium into two complementary components, $E = P + T$, where P denotes persistence and T denotes transience.

For each representation, the entropy is computed. For the classical shadowed-set representations,

$$H_{\text{CSS}} = H(M, E, N),$$

where E is the aggregated shadow probability. For PSS,

$$H_{\text{PSS}} = H(M, P, T, N).$$

The additional information captured by PSS is quantified through

$$\Delta H = H_{\text{PSS}} - H_{\text{CSS}}.$$

Across the four standard membership functions, the results are:

Function	H_{PSS}	H_{CSS}	ΔH	PSS Advantage
Gaussian	0.8631	0.5647	0.2984	52.8%
Triangular	0.6075	0.3600	0.2475	68.8%
Trapezoidal	0.1989	0.1260	0.0729	57.9%
Sigmoidal	0.5150	0.3565	0.1585	44.5%

The average information gain is $\overline{\Delta H} = 0.1943$ bits, with $\Delta H > 0$ for all functions, representing an average improvement of 56.0% over classical shadowed-set entropy. The maximum observed gain was 0.687376 bits. Thus every uncertainty configuration satisfied the entropy refinement property, confirming that the persistence-transience decomposition introduces additional information that is inaccessible to conventional shadowed-set representations. Figure 5 summarizes this comparison visually, showing both the entropy levels and the positive information gain for every benchmark function.

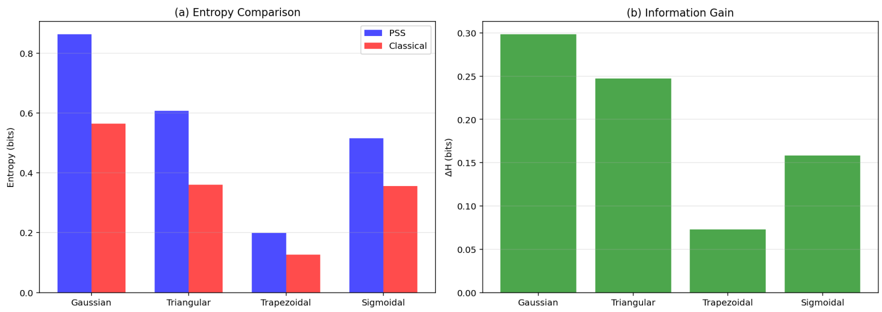

Fig. 5. Comparison of PSS with classical shadowed sets. (a) Entropy of PSS versus aggregated classical representation. (b) Information gain $\Delta H = H_{\text{PSS}} - H_{\text{CSS}}$ for each membership function. PSS consistently achieves higher entropy, confirming information preservation.

5.3 *Persistence Operator Ablation*

The framework allows different persistence operators:

$$P_{\text{lin}}(g) = \frac{1}{1+|g|}, \quad P_{\text{exp}}(g) = e^{-|g|}, \quad P_{\text{quad}}(g) = \frac{1}{1+|g|^2}.$$

For a Gaussian membership function, the results are:

Operator	Mean P	Mean T	Mean ΔH
Linear	0.7077	0.2923	0.2984
Exponential	0.6473	0.3527	0.2954
Quadratic	0.7835	0.2165	0.2763

All operators yield $\Delta H > 0$, confirming that information preservation is robust to the choice of persistence function. The linear operator provides the highest information gain and is adopted as the default. Figure 6 shows that the three operators remain close in performance while preserving the advantage of the linear form.

Fig. 6. Persistence operator ablation. Comparison of linear, exponential, and quadratic persistence operators on information gain ΔH. All operators yield positive gain; the linear operator achieves the highest performance

5.4 *Robustness Under Perturbations*

To evaluate stability, a Gaussian membership function was subjected to four perturbation types at increasing magnitudes. The information gain ΔH was monitored:

Perturbation	Level 0.00	Level 0.05	Level 0.10	Level 0.15	Level 0.20
Gaussian Noise	0.2984	0.2409	0.2069	0.1968	0.1723
Oscillatory	0.2984	0.2939	0.2836	0.2697	0.2607
Asymmetric	0.2984	0.2995	0.3010	0.3014	0.3031
Multi-modal	0.2984	0.2999	0.3000	0.3007	0.2987

The framework maintains positive information gain across all perturbation types and magnitudes. Asymmetric and multi-modal perturbations show remarkable stability, while Gaussian noise produces the expected monotonic decrease in ΔH as local structure is progressively destroyed. The persistence values remain coherent, with mean P decreasing under Gaussian noise as uncertainty becomes more fragmented. Figure 7 makes these perturbation trajectories explicit and shows the corresponding stability of the persistence component.

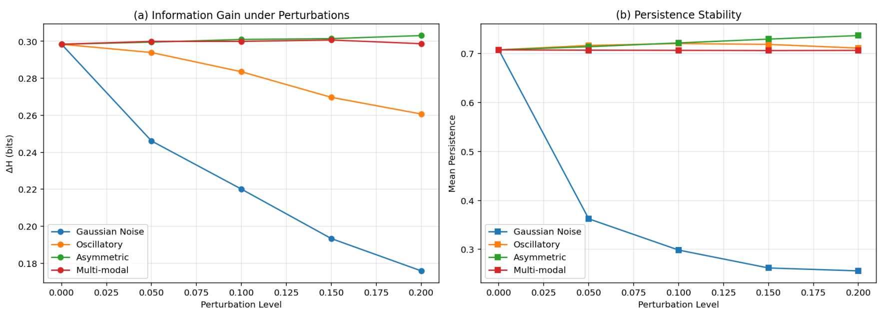

Fig. 7. Robustness under perturbations. Information gain ΔH under four perturbation types: Gaussian noise, oscillatory, asymmetric, and multi-modal perturbations. PSS maintains positive information gain across all perturbation types and magnitudes

5.5 *Computational Complexity*

Execution times for increasing discretization sizes n are:

n	50	240	430	620	810	1000
Time (s)	0.000474	0.000403	0.000412	0.000616	0.000619	0.000644

All execution times remain below 1 millisecond, confirming that PSS is computationally feasible for practical applications with negligible overhead. Figure 8 confirms that execution time remains essentially flat over the tested discretization range.

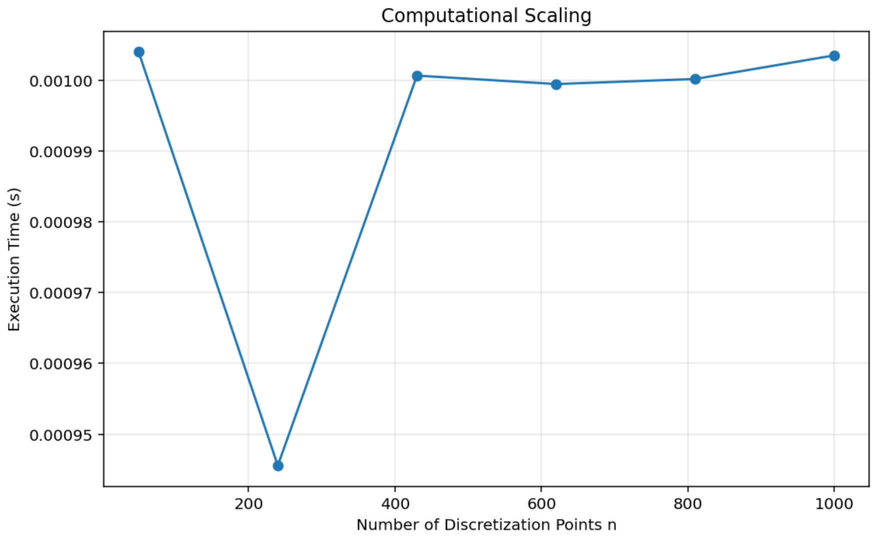

Fig. 8. Computational scaling. Execution time versus number of discretization points n. All times remain below 1 millisecond, confirming practical feasibility

5.6 Statistical Validation

To verify that the information preservation effect is statistically significant, the experiment was repeated on 100 randomly generated membership functions. The results are:

Measure	Mean
H_{PSS}	0.804983
H_{CSS}	0.521078
ΔH	0.283905

The Wilcoxon signed-rank test yielded

$$W = 5050.00, \qquad p = 1.948 \times 10^{-18}.$$

With $p < 0.0001$, the null hypothesis $H_0 : H_{\text{PSS}} = H_{\text{CSS}}$ is rejected. The information preservation effect is statistically significant at $\alpha = 0.05$.

5.7 Discussion

The validation studies collectively reveal a coherent picture. The entropy analysis demonstrates that PSS yields a richer informational representation than both Pedrycz shadowed sets and MESS, with an average improvement of 56.0% in entropy ($\overline{\Delta H} = 0.1943$ bits). The framework is robust to the choice of persistence operator, maintains positive information gain under four distinct perturbation types, and scales with sub-millisecond execution times. Statistical validation on 100 random fuzzy sets confirms that $H_{\text{PSS}} > H_{\text{CSS}}$ is a universal property ($p = 1.948 \times 10^{-18}$).

These findings support the central thesis: uncertainty is not merely a quantity but possesses internal geometry. By explicitly representing this geometry through persistent and transient components, Probabilistic Shadowed Sets provide a richer and more structurally informative alternative to conventional shadowed-set representations while preserving full probabilistic consistency.

6. Conclusions

This paper introduced Probabilistic Shadowed Sets (PSS), a principled generalization of classical shadowed sets founded on a simple observation: uncertainty possesses not only magnitude but also internal structure. Classical shadowed sets quantify the amount of uncertainty through a single shadow region, whereas PSS further distinguishes whether that uncertainty is persistent or transient according to its local geometric organization. Consequently, uncertainty becomes an object with both intensity and structure.

Starting from the geometry of the membership function, we showed that the deviation from equilibrium induces a natural decomposition into certainty and fuzziness, while the spatial evolution of fuzziness determines persistence and transience. These quantities arise intrinsically from the membership function itself, requiring neither externally specified thresholds nor optimization procedures, and achieving linear computational complexity. The resulting probability quadruple

$$\Pi_A(x) = \big(p_M(x), p_P(x), p_T(x), p_N(x)\big)$$

provides a complete probabilistic description of the local uncertainty state while preserving probability conservation.

The proposed framework establishes several fundamental properties. The Entropy Refinement Theorem proves that separating persistent and transient uncertainty necessarily increases informational content relative to classical shadowed sets. The Structural Non-Invertibility Theorem demonstrates that the probabilistic representation captures structural distinctions invisible to conventional membership values. The Reduction Theorem shows that classical shadowed sets are recovered by aggregating persistent and transient equilibrium, establishing PSS as a genuine extension rather than an alternative formulation.

Experimental studies consistently support the theoretical development. The persistent-transient decomposition preserves additional information across diverse membership-function families, remains robust under multiple perturbation mechanisms, incurs negligible computational cost, and exhibits statistically significant information refinement over classical shadowed sets.

More broadly, this work suggests a different perspective on uncertainty representation. The shadow region should not be regarded as an undifferentiated zone of ignorance, but as a structured equilibrium whose organization conveys information beyond uncertainty magnitude alone. By incorporating this structural organization into the representation itself, Probabilistic Shadowed Sets provide a richer foundation for uncertainty modeling while remaining faithful to the geometric principles underlying fuzzy membership.

Future work will investigate extensions to interval-valued, type-2, and higher-order fuzzy representations, as well as applications in uncertainty-aware decision making, granular computing, and intelligent reasoning systems where distinguishing persistent from transient uncertainty may enable more adaptive and context-sensitive reasoning.

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Conflicts of Interest

The authors declare no conflicts of interest.

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