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Fuzzy, Neutrosophic, and Plithogenic Approaches to Project Scheduling, Multischeduling, and Iterative Multischeduling

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ABSTRACT

This paper develops a unified framework for project scheduling under uncertainty using Iterative MultiStructures. We introduce Fuzzy, Neutrosophic, and Plithogenic Project Iterative MultiScheduling, in which schedules are composed across hierarchical levels as finite multisets and evaluated after canonical flattening and selection. Using componentwise extensions of the sum and max operators, we propagate uncertain activity times and derive fuzzy expectations and credibility constraints, together with their neutrosophic and plithogenic counterparts. We prove that the schedule space forms an Iterative MultiStructure and that the proposed models strictly generalize classical fuzzy, neutrosophic, and plithogenic project scheduling, as well as their non-iterative multischeduling variants. The framework supports precedence networks and multi-criteria costs and admits exact reductions under natural embeddings.

1. Introduction

Real-world decision problems are pervaded by uncertainty, vagueness, and incomplete information. To model such phenomena, a wide spectrum of generalized set-theoretic formalisms has been developed. Among the most influential is the *fuzzy set*, which assigns to each element a membership grade in the unit interval $[0, 1]$, thereby replacing a strict yes/no boundary by graded inclusion [1]. Many refinements and companion notions have been proposed, including bipolar fuzzy sets [2], shadowed sets [3], and m -polar (multipolar) fuzzy sets [4]. In addition, *neutrosophic sets* extend fuzzy sets by explicitly incorporating an *indeterminacy* component, which enables a more flexible representation of statements that are neither entirely true nor entirely false [5, 6]. More recently, *plithogenic sets* have been introduced to represent membership through explicit attributes together with a contradiction (or dissimilarity) mapping between attribute values, thereby capturing multi-attribute reasoning in the presence of conflicting evidence [7–9]. We present a concise comparison of fuzzy, neutrosophic, and plithogenic set frameworks in Table 1.

In this paper, we recall the notions required for our development and emphasize the extensions that are most relevant to the scheduling setting. Project scheduling is a foundational optimization and decision task in operations research and management science: one seeks start and finish times for a set of interdependent activities so as to satisfy precedence relations and resource constraints while op-

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Table 1
Concise comparison of fuzzy, neutrosophic, and plithogenic set frameworks.

Aspect	Fuzzy set	Neutrosophic set	Plithogenic set
Element description	One grade $\mu(x) \in [0, 1]$	Triple $(T(x), I(x), F(x))$ (truth/indeterminacy/falsity degrees)	Attribute–value memberships $\mu_{a=v}(x)$ together with a contradiction map $c(v_i, v_j)$ between attribute values
Uncertainty captured	Vagueness / partial truth	Vagueness plus explicit indeterminacy and inconsistency	Multi-attribute uncertainty with explicit contradictions among attribute values
Role of attributes	Usually implicit (single criterion)	Usually implicit (three components)	Explicit attribute set and value domains; evaluation is attribute-driven
Component relations	Single component; complement often taken as $1 - \mu(x)$ (optional)	Components treated as independent; constraints depend on the chosen variant	Contradiction degrees modulate how attribute memberships are fused
Aggregation idea	t-norm/t-conorm based fusion, fuzzy implication, defuzzification	Componentwise fusion of T, I, F with suitable norms/conorms	Contradiction-aware fusion: operators/weights depend on $c(\cdot, \cdot)$
Best suited for	Graded concepts under a single (or few) criteria	Settings with “unknown” and/or conflicting evidence	Decision problems where attribute-value preferences may conflict

timizing overall objectives such as duration, cost, and quality. A substantial literature studies project scheduling from fuzzy and neutrosophic perspectives. In contrast, scheduling models grounded in plithogenic sets have received comparatively less attention, and the situation is even more limited for complex paradigms such as *project multischeduling* and *iterative multischeduling*, in which multiple candidate schedules must be generated, combined, and re-evaluated across successive planning layers or cycles. Nevertheless, practical project management frequently requires exactly this type of complex handling: decision makers compare competing schedule proposals, revise them as information changes, and coordinate plans across hierarchical levels.

To bridge this gap, this paper develops a unified framework for project scheduling under uncertainty using *Iterative MultiStructures*. We introduce *Fuzzy*, *Neutrosophic*, and *Plithogenic Project Iterative MultiScheduling*, where schedules are composed across hierarchical levels as finite multisets and assessed via a canonical *flattening* and *selection* mechanism. At a single level, *multischeduling* maintains multiple candidate schedules in parallel and combines proposals through specified multi-operations, enabling systematic evaluation, selection, and robustness analysis. At multiple levels (or over repeated planning cycles), *iterative multischeduling* repeats this process hierarchically: combined multisets are flattened, information is updated, and schedules are reselected for execution. This perspective provides a principled way to represent and compute with collections of competing plans while remaining compatible with fuzzy, neutrosophic, and plithogenic uncertainty semantics. For reference, a concise comparison of project scheduling, project multischeduling, and project iterative multischeduling is provided in Table 2.

Table 2
Concise comparison of project scheduling, project multischeduling, and project iterative multischeduling.

Aspect	Project scheduling	Project multischeduling	Project iterative multi-scheduling
Primary output	One executable schedule (single plan).	A finite set/multiset of candidate schedules maintained in parallel.	A hierarchy/sequence of multisets of schedules across levels or cycles, with a selected schedule at each stage.
Core idea	Optimize a single schedule under precedence/resource constraints.	Generate/keep multiple alternatives; combine proposals via multi-operations; then evaluate/select.	Repeat multischeduling across hierarchical levels (e.g., WBS layers) or repeated replanning cycles; flatten, update, and reselect.
Where uncertainty enters	Parameters of one model (durations, costs, resources) may be uncertain.	Uncertainty affects both parameters and <i>the space of plausible schedules</i> ; diversity is explicit.	Uncertainty evolves over time/levels; information updates and cross-level composition are explicit.
Combination mechanism	Not essential (single solution).	Explicit merge/aggregation of candidate schedules (e.g., multiset union, weighted fusion, conflict-aware operators).	Composition of multisets across levels, followed by canonical <i>flattening</i> to comparable candidates.
Evaluation and decision	Choose the best (or Pareto-optimal) schedule once.	Score many candidates; select one or keep a shortlist; analyze robustness/sensitivity across candidates.	At each level/cycle: evaluate flattened candidates, select, update data/constraints, and propagate decisions forward.
Typical use case	Stable environment; one plan is sufficient.	Multiple stakeholders/scenarios; need contingency plans and robustness comparisons.	Large projects with layered planning and frequent replanning; rolling-wave management and hierarchical integration.
Computational profile	Solve one optimization problem.	Solve and compare many candidate problems (parallelizable); requires diversity management.	Nested/recursive process; requires level-wise book-keeping, flattening, and consistent update rules.

2. Preliminaries

This section introduces the fundamental concepts and definitions that underpin the discussions in this paper. Throughout, all sets are assumed to be finite.

2.1 Fuzzy Set and Neutrosophic Set

A fuzzy set assigns to each element a grade in the interval $[0, 1]$, thereby representing uncertainty by degrees rather than by a strict binary split [10]. Related concepts include the Bipolar Fuzzy Set [2], the Spherical Fuzzy Set [11, 12], the Picture Fuzzy Set [13], the Shadowed Set [3], and the m -polar Fuzzy Set [4]. Below we recall the relevant notions, together with their extensions.

Definition 2.1 (Fuzzy Set). [1, 14] Let Y be a nonempty universe. A *fuzzy set* on Y is a mapping $\tau : Y \rightarrow [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset $\delta : Y \times Y \rightarrow [0, 1]$. We say that δ is a *fuzzy relation on τ* if, for all $y, z \in Y$,

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\}.$$

Example 2.2 (Fuzzy set in project management: uncertain task duration). Let $Y = \mathbb{R}_{\geq 0}$ denote possible durations (days) of a software testing task. Model the expert estimate “about 10 days, realistically between 8 and 14” by the triangular fuzzy set $\tau : Y \rightarrow [0, 1]$ with

$$\tau(t) = \begin{cases} 0, & t < 8, \\ \frac{t-8}{10-8}, & 8 \leq t \leq 10, \\ \frac{14-t}{14-10}, & 10 < t \leq 14, \\ 0, & t > 14. \end{cases}$$

Here $\tau(9) = 0.5$ expresses a moderate plausibility of finishing in 9 days, while $\tau(12) = 0.5$ quantifies a similar plausibility for 12 days. Such fuzzy durations can be propagated through a precedence network via Zadeh’s extension principle to compute fuzzy earliest/finish times and support time–cost tradeoffs under uncertainty.

Neutrosophic sets extend fuzzy sets by explicitly incorporating *indeterminacy*, thus covering statements that are neither entirely true nor entirely false, and offering a flexible representation of ambiguity [5]. Related concepts include the Bipolar Neutrosophic Set, the Hesitant Neutrosophic Set [15, 16], Neutrosophic Rough Set [17], the Neutrosophic Cubic Set [18, 19], the Complex Neutrosophic Set [20, 21], and the Interval-Valued Neutrosophic Set [22, 23]. Their formal specification is as follows.

Definition 2.3 (Neutrosophic Set). [5] Let X be a nonempty set. A *Neutrosophic Set (NS)* A on X is determined by three functions

$$T_A : X \rightarrow [0, 1], \quad I_A : X \rightarrow [0, 1], \quad F_A : X \rightarrow [0, 1],$$

where, for each $x \in X$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.4 (Neutrosophic set in project management: on-time supplier delivery). Let $X = \{A, B\}$ be two hardware suppliers and consider the statement “deliver the batch by Day 20.” Represent each supplier by a single-valued neutrosophic triple $(T, I, F) \in [0, 1]^3$ capturing, respectively, the degrees of truth, indeterminacy, and falsity for that statement:

$$\text{Supplier A : } (T, I, F) = (0.75, 0.15, 0.10), \quad \text{Supplier B : } (T, I, F) = (0.55, 0.30, 0.15).$$

Supplier A has stronger supporting evidence ($T = 0.75$) and less conflict ($F = 0.10$), with moderate ambiguity ($I = 0.15$); Supplier B is less convincing and more ambiguous. These triples let the project manager rank or threshold suppliers while explicitly accounting for conflicting information and residual indeterminacy in the schedule risk assessment.

Neutrosophic Sets have been extensively studied in the field of project management [24–26]. These studies demonstrate how neutrosophic logic effectively models uncertainty, inconsistency, and incompleteness in project evaluation, scheduling, and decision-making processes.

2.2 Plithogenic Set

A plithogenic set [7–9] represents elements via membership driven by explicit attributes together with a contradiction mapping between attribute values, thereby subsuming and extending the frameworks of Fuzzy [1], Intuitionistic [27], and Neutrosophic sets [6, 28].

Definition 2.5 (Plithogenic Set). [29] Let S be a universe and $P \subseteq S$ a nonempty subset. A *plithogenic set* is a 5-tuple

$$PS = (P, v, Pv, pdf, pCF),$$

with the following ingredients:

- v — a chosen attribute;
- Pv — the value domain of v ;
- $pdf : P \times Pv \rightarrow [0, 1]^s$ — the *degree of appurtenance* (DAF);*
- $pCF : Pv \times Pv \rightarrow [0, 1]^t$ — the *degree of contradiction* (DCF).

For every $a, b \in Pv$, the DCF satisfies

$$\text{reflexivity: } pCF(a, a) = 0, \quad \text{symmetry: } pCF(a, b) = pCF(b, a).$$

Here $s, t \in \mathbb{N}$ are, respectively, the appurtenance and contradiction dimensions.

Example 2.6 (Plithogenic set in project management: sprint plan under attribute contradiction). Let $P = \{\text{Plan}_1, \text{Plan}_2\}$ be two sprint plans for a six-week release. Choose the attribute $v = \text{“risk level”}$ with value domain $Pv = \{\text{low, medium, high}\}$. Use a (scalar) degree-of-appurtenance $pdf : P \times Pv \rightarrow [0, 1]$ to encode how strongly each plan fits “feasible within 6 weeks” under each risk value:

	low	medium	high
Plan ₁	0.70	0.40	0.10
Plan ₂	0.50	0.60	0.30

Specify a symmetric degree-of-contradiction $pCF : Pv \times Pv \rightarrow [0, 1]$ with

$$pCF(\text{low, low}) = pCF(\text{medium, medium}) = pCF(\text{high, high}) = 0,$$

$$pCF(\text{low, medium}) = 0.30, \quad pCF(\text{medium, high}) = 0.60, \quad pCF(\text{low, high}) = 1.00,$$

and $pCF(a, b) = pCF(b, a)$. The pair (pdf, pCF) constitutes a plithogenic description of feasibility: attributes with stronger mutual contradiction (e.g., “low” vs. “high” risk) are penalized more by the plithogenic aggregation. A manager can aggregate these values to obtain an effective feasibility for each plan that respects both membership strengths and inter-value contradictions, then select the plan with the higher effective feasibility.

*In the literature, DAF is modeled in several equivalent ways (e.g., powerset-valued or vector-valued). We adopt the standard $[0, 1]^s$ form; see [30].

2.3 Multi-Structure

A Multi-Structure replaces classical operations with maps from tuples to finite multisets, enabling multiple outputs per input tuple flexibly simultaneously [31].

Definition 2.7 (Finite Multiset). Let H be a nonempty set. A *finite multiset* on H is a function

$$m : H \longrightarrow \mathbb{N}_0$$

with finite support $\{x \in H \mid m(x) > 0\}$. We denote by $\mathcal{M}(H)$ the collection of all such finite multisets on H . Equivalently, an element of $\mathcal{M}(H)$ can be written as $\{x_1^{k_1}, x_2^{k_2}, \dots, x_r^{k_r}\}$, where each $x_i \in H$ and $k_i = m(x_i) \in \mathbb{N}$.

Definition 2.8 (MultiOperation). Let H be a nonempty set and fix an integer $m \geq 1$. A *multi-operation* of arity m on H is a map

$$\begin{aligned} \#^{(m)} : H^m &\longrightarrow \mathcal{M}(H), \\ (x_1, \dots, x_m) &\mapsto \#^{(m)}(x_1, \dots, x_m) \in \mathcal{M}(H). \end{aligned}$$

Thus, instead of producing a single element of H , a multi-operation assigns a finite multiset of elements of H .

Definition 2.9 (MultiStructure). A *MultiStructure* is a pair

$$\mathcal{MS} = (H, \{\#^{(m)} : H^m \rightarrow \mathcal{M}(H)\}_{m \in \mathcal{I}}),$$

where H is a nonempty carrier set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ indexes a family of multi-operations of various arities. No further axioms are imposed unless specified.

Example 2.10 (Multi-Structure in everyday project planning: combining schedule proposals). Let $H = \mathbb{R}_{\geq 0}^3$ denote start times (in days) for three tasks. Given two proposed schedules $x = (0, 3, 6)$ and $y = (1, 4, 5)$, define the binary multi-operation $\#^{(2)} : H^2 \rightarrow \mathcal{M}(H)$ by three concrete *combiners*:

$$A_{\min}(x, y) = \min_{\text{coord}}(x, y), \quad A_{\max}(x, y) = \max_{\text{coord}}(x, y), \quad A_{\text{avg}}(x, y) = \frac{1}{2}(x + y).$$

Then

$$\#^{(2)}(x, y) = \{A_{\min}(x, y), A_{\max}(x, y), A_{\text{avg}}(x, y)\} = \{(0, 3, 5), (1, 4, 6), (0.5, 3.5, 5.5)\} \in \mathcal{M}(H).$$

Thus a *single* pair of inputs produces a *finite multiset* of candidate schedules: an aggressive timeline (min), a conservative one (max), and a compromise (avg). This models a common meeting where two teams propose timelines and the planner keeps multiple viable variants simultaneously.

2.4 Iterative Multi-Structure

An Iterative Multi-Structure extends multiset operations across levels, combining multisets of multisets iteratively through k hierarchical stages in layered aggregation [32].

Definition 2.11 (Iterative Multi-Structure of Order k). Let H be a nonempty set and fix an integer $k \geq 1$. Define iteratively the *multiset powersets*

$$\mathcal{M}^0(H) = H, \quad \mathcal{M}^{i+1}(H) = \mathcal{M}(\mathcal{M}^i(H)), \quad i = 0, 1, \dots, k-1,$$

where $\mathcal{M}(X)$ denotes the collection of finite multisets on X (Definition 2.7). Let $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ index a family of arities. An *Iterative Multi-Structure of order k* is a tuple

$$\mathcal{IMS}^{(k)} = (H, \{\#^{(m,i)} : (\mathcal{M}^i(H))^m \longrightarrow \mathcal{M}^{i+1}(H)\}_{m \in \mathcal{I}, 0 \leq i < k}),$$

where for each $i = 0, \dots, k-1$ and each $m \in \mathcal{I}$,

$$\#^{(m,i)}(x_1, \dots, x_m) \in \mathcal{M}^{i+1}(H), \quad x_j \in \mathcal{M}^i(H).$$

Thus $\#^{(m,0)}$ is an ordinary Multi-Structure operation on H , $\#^{(m,1)}$ combines multisets of multisets, and so on, up to level k .

Example 2.12 (Iterative Multi-Structure in practice: two-level portfolio scheduling). Consider a company release involving two departments, Marketing (M) and Engineering (E). Let $H = \mathbb{R}_{\geq 0}^2$ be start times for two key milestones. Each department supplies two proposals:

$$x_1^M = (0, 2), \quad x_2^M = (1, 1.5), \quad x_1^E = (0.5, 1.8), \quad x_2^E = (0.2, 2.2).$$

Level 0 (within-department mixing). Define $\#^{(2,0)} : H^2 \rightarrow \mathcal{M}(H)$ using the three combiners from the previous example. For Marketing,

$$M_0 = \#^{(2,0)}(x_1^M, x_2^M) = \{(0, 1.5), (1, 2), (0.5, 1.75)\} \in \mathcal{M}(H).$$

For Engineering,

$$E_0 = \#^{(2,0)}(x_1^E, x_2^E) = \{(0.2, 1.8), (0.5, 2.2), (0.35, 2.0)\} \in \mathcal{M}(H).$$

Level 1 (cross-department aggregation). Define $\#^{(2,1)} : (\mathcal{M}(H))^2 \rightarrow \mathcal{M}(\mathcal{M}(H))$ by the combiner

$$A_{\max}^{(1)}(M_0, E_0) = \{ \max_{\text{coord}}(m, e) : m \in \text{supp}(M_0), e \in \text{supp}(E_0) \},$$

collecting every coordinatewise-feasible portfolio schedule. Hence

$$\#^{(2,1)}(M_0, E_0) = \{ A_{\max}^{(1)}(M_0, E_0) \} \in \mathcal{M}(\mathcal{M}(H)),$$

which is a *multiset of multisets* (each inner multiset is a finite set of combined schedules). Flattening by Unnest (as defined elsewhere) yields a finite multiset of portfolio schedules in H . This two-level construction is an *Iterative Multi-Structure of order $k = 2$* : level 0 mixes proposals *within* departments; level 1 aggregates department-level results into company-level candidates, reflecting a common real workflow in release planning.

2.5 Project Scheduling and Fuzzy Project Scheduling

Project scheduling assigns start/finish times to interdependent activities on a precedence network to optimize overall duration, cost, and resource usage [33]. Fuzzy project scheduling represents uncertain activity durations by fuzzy numbers, optimizing network objectives using expected values or credibility-constrained mathematical approaches [34–36].

Definition 2.13 (Project Scheduling (deterministic)). Let a project be represented by a directed acyclic graph $G = (V, A)$ with source 1 and sink $n+1$. Each arc $(i, j) \in A$ is an activity of deterministic duration $d_{ij} > 0$ and cost $c_{ij} \geq 0$. For decision vector $x = (x_1, \dots, x_n)$ (e.g., release/start or funding times), define node start times $T_i(x, d)$ by

$$T_1(x, d) = x_1, \quad T_i(x, d) \geq x_i, \quad T_i(x, d) \geq \max_{(k,i) \in A} \{T_k(x, d) + d_{ki}\} \quad (i = 2, \dots, n),$$

and project completion time

$$T(x, d) = \max_{(k,n+1) \in A} \{T_k(x, d) + d_{k,n+1}\}.$$

If financing accrues at rate $r \geq 0$, the total cost is

$$C(x, d) = \sum_{(i,j) \in A} c_{ij} (1 + r)^{\lceil T(x,d) - x_i \rceil}.$$

The (deterministic) project scheduling problem is

$$\min_x C(x, d) \quad \text{s.t.} \quad T(x, d) \leq T_0, \quad x \geq 0 \quad (\text{integer or continuous as modeled}).$$

This captures the classical tradeoff between completion time and total cost.

Example 2.14 (Project Scheduling in practice: office renovation with a critical path). Consider an office renovation with tasks arranged as a DAG $G = (V, A)$:

$$V = \{1, 2, 3, 4, 5\} \text{ (source 1, sink 5)}, \quad A = \{(1, 2), (2, 3), (2, 4), (3, 5), (4, 5)\}.$$

Interpretation: (1, 2) “permit approval”, (2, 3) “demolition”, (2, 4) “electrical & HVAC”, (3, 5) “finishes”, (4, 5) “inspection & handover”. Durations (days) and costs (k\$) are deterministic:

$$\begin{aligned} d_{1,2} = 4, \quad c_{1,2} = 100; & \quad d_{2,3} = 3, \quad c_{2,3} = 120; & \quad d_{2,4} = 2, \quad c_{2,4} = 80; \\ d_{3,5} = 2, \quad c_{3,5} = 150; & \quad d_{4,5} = 5, \quad c_{4,5} = 90. \end{aligned}$$

Choose start vector $x = (x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$. Then

$$T_2 = \max\{x_2, T_1 + d_{1,2}\} = \max\{0, 0 + 4\} = 4, \quad T_3 = \max\{x_3, T_2 + d_{2,3}\} = \max\{0, 4 + 3\} = 7,$$

$$T_4 = \max\{x_4, T_2 + d_{2,4}\} = \max\{0, 4 + 2\} = 6,$$

$$T(x, d) = \max\{T_3 + d_{3,5}, T_4 + d_{4,5}\} = \max\{7 + 2, 6 + 5\} = 11.$$

Thus the critical path is $1 \rightarrow 2 \rightarrow 4 \rightarrow 5$ with project completion time 11 days. If financing accrues at $r = 0.05$, the total cost is

$$C(x, d) = \sum_{(i,j) \in A} c_{ij} (1 + r)^{\lceil T(x,d) - x_i \rceil} = (1.05)^{11} (100 + 120 + 80 + 150 + 90).$$

This mirrors a real renovation: parallel trades (demolition vs. MEP work) and a longer inspection stage determine the finish date and the capitalized cost.

Definition 2.15 (Fuzzy Project Scheduling). Assume each activity duration is a fuzzy quantity $\tilde{n}_{ij}$ (e.g., a trapezoidal fuzzy number) on the same network $G = (V, A)$. Let $T_i(x, \tilde{n})$, $T(x, \tilde{n})$ and $C(x, \tilde{n})$ be the induced fuzzy start time, completion time, and cost. Using the credibility measure $\text{Cr}\{\cdot\}$ and fuzzy expectation $\mathbb{E}[\cdot]$, three concise formulations are standard:

$$\begin{aligned} \text{(EVM)} \quad & \min_x \mathbb{E}[C(x, \tilde{n})] \\ & \text{s.t.} \quad \mathbb{E}[T(x, \tilde{n})] \leq T_0, \quad x \geq 0; \end{aligned}$$

$$\begin{aligned} \text{(\alpha-cost)} \quad & \min_{x, C_0} C_0 \\ & \text{s.t.} \quad \text{Cr}\{C(x, \tilde{n}) \leq C_0\} \geq \alpha, \\ & \quad \text{Cr}\{T(x, \tilde{n}) \leq T_0\} \geq \beta, \quad x \geq 0; \end{aligned}$$

$$\begin{aligned} \text{(Credibility max)} \quad & \max_x \text{Cr}\{C(x, \tilde{n}) \leq C_0\} \\ & \text{s.t.} \quad \text{Cr}\{T(x, \tilde{n}) \leq T_0\} \geq \alpha, \quad x \geq 0. \end{aligned}$$

These formulations are widely used to balance time and cost under fuzzy activity times via credibility/chance constraints and fuzzy expectation.

Example 2.16 (Fuzzy Project Scheduling in practice: software release under uncertain task times). Reuse the same network $G = (V, A)$ but model activity durations by triangular fuzzy numbers (TFNs):

$$\begin{aligned}\tilde{n}_{1,2} &= \text{TFN}(3.5, 4, 4.5), & \tilde{n}_{2,3} &= \text{TFN}(2.5, 3, 3.5), & \tilde{n}_{2,4} &= \text{TFN}(1.5, 2, 2.5), \\ \tilde{n}_{3,5} &= \text{TFN}(1.5, 2, 2.5), & \tilde{n}_{4,5} &= \text{TFN}(4.5, 5, 5.5).\end{aligned}$$

These represent, for a software release, uncertain durations for “requirements approval”, “refactor legacy”, “infrastructure & CI”, “UI polish”, and “security review”. Using Zadeh’s extension principle, the fuzzy completion time $\tilde{T}(x, \tilde{n})$ is induced from path sums and the network max. For planning, apply centroid defuzzification $\text{cen}(\text{TFN}(a, b, c)) = (a+b+c)/3$ to approximate expected durations:

$$\mathbb{E}[\tilde{n}_{1,2}] = 4, \quad \mathbb{E}[\tilde{n}_{2,3}] = 3, \quad \mathbb{E}[\tilde{n}_{2,4}] = 2, \quad \mathbb{E}[\tilde{n}_{3,5}] = 2, \quad \mathbb{E}[\tilde{n}_{4,5}] = 5.$$

Hence the expected path lengths are

$$\mathbb{E}[1 \rightarrow 2 \rightarrow 3 \rightarrow 5] = 4 + 3 + 2 = 9, \quad \mathbb{E}[1 \rightarrow 2 \rightarrow 4 \rightarrow 5] = 4 + 2 + 5 = 11,$$

so the latter remains the (expected) critical path with $\mathbb{E}[\tilde{T}] = 11$. If a budget accrual uses the same $r = 0.05$, an expected-cost proxy is

$$\mathbb{E}[C(x, \tilde{n})] \approx (1.05)^{\mathbb{E}[\tilde{T}]} (100 + 120 + 80 + 150 + 90) = (1.05)^{11} \cdot 540.$$

Decision-makers can also impose credibility constraints, e.g. require $\text{Cr}\{\tilde{T} \leq T_0\} \geq \beta$, to certify release dates at a chosen confidence level while comparing candidate schedules.

3. Main Results

This section presents the main results of the paper.

3.1 Neutrosophic Project Scheduling

Neutrosophic Project Scheduling models activity durations as Single-Valued Neutrosophic Numbers (SVNNs), propagating truth, indeterminacy, and falsity through the project network under credibility and expectation constraints. Several research works have investigated the Neutrosophic Set from the perspective of Project Scheduling, highlighting its capability to handle incomplete, inconsistent, and uncertain temporal information in practical project management scenarios (cf. [37–39]).

Definition 3.1 (Neutrosophic Project Scheduling (NPS)). Let the project network be a finite DAG $G = (V, A)$ with source 1 and sink $n+1$. For each activity $(i, j) \in A$, its duration is modeled by a *single-valued neutrosophic number*

$$\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad T_{ij}, I_{ij}, F_{ij} : \mathbb{R}_{\geq 0} \rightarrow [0, 1],$$

where $T_{ij}(x)$, $I_{ij}(x)$, and $F_{ij}(x)$ denote the degrees of truth, indeterminacy, and falsity for the statement “the duration equals x ”. For a decision vector $x = (x_1, \dots, x_n)$ of planned release/start times, define the *neutrosophic start times* $\tilde{T}_i(x, \tilde{n})$ and the *neutrosophic completion time* $\tilde{T}(x, \tilde{n})$ as single-valued neutrosophic numbers on $\mathbb{R}_{\geq 0}$ obtained by Zadeh’s extension principle applied componentwise to the operators $(u, v) \mapsto u+v$ and $(u, v) \mapsto \max\{u, v\}$:

$$\tilde{T}_1(x, \tilde{n}) = \tilde{x}_1, \quad \tilde{T}_i(x, \tilde{n}) \succeq \tilde{x}_i, \quad \tilde{T}_i(x, \tilde{n}) \succeq \max_{(k,i) \in A} (\tilde{T}_k(x, \tilde{n}) + \tilde{n}_{ki}),$$

$$\tilde{T}(x, \tilde{n}) = \max_{(k,n+1) \in A} (\tilde{T}_k(x, \tilde{n}) + \tilde{n}_{k,n+1}).$$

(Here “+” and “max” act via the extension principle on each of T, I, F ; the symbol $\succeq$ indicates the induced partial order on neutrosophic numbers.) Let Cr_N be a *neutrosophic credibility* that assigns to each neutrosophic event E a value in $[0, 1]$, and let $\mathbb{E}_N[\cdot]$ be a compatible neutrosophic expectation functional. Given targets $T_0 > 0$, $\alpha, \beta \in (0, 1]$, and a (possibly neutrosophic) cost functional $\tilde{C}(x, \tilde{n})$, the *NPS problem* is any of the following equivalent chance/expectation formulations:

$$\textbf{(N-cred)} \quad \min_{x, C_0} C_0 \quad \text{s.t. } \text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq T_0\} \geq \beta, \quad \text{Cr}_N\{\tilde{C}(x, \tilde{n}) \leq C_0\} \geq \alpha, \quad x \geq 0;$$

$$\textbf{(N-exp)} \quad \min_x \mathbb{E}_N[\tilde{C}(x, \tilde{n})] \quad \text{s.t. } \mathbb{E}_N[\tilde{T}(x, \tilde{n})] \leq T_0, \quad x \geq 0.$$

Example 3.2 (Neutrosophic Project Scheduling — Hospital Renovation). A hospital renovates an operating theatre over three serial activities on a DAG $G = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (3, 4)\}$: demolition (1, 2), HVAC installation (2, 3), and inspection (3, 4). Each activity duration is modeled by a single-valued neutrosophic number $\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij})$ (time in days). For demolition,

$$T_{12}(x) = \max\{0, 1 - |x - 5|\}, \quad I_{12}(x) = 0.2 \text{ for } x \in [4, 6], \quad 0 \text{ elsewhere}, \quad F_{12}(x) = 1 - T_{12}(x),$$

so the most plausible duration is 5 days with small indeterminacy (permits, hidden pipes). Similarly,

$$T_{23}(x) = \max\{0, 1 - |x - 6|\}, \quad I_{23}(x) = 0.15 \text{ on } [5, 7], \quad F_{23}(x) = 1 - T_{23}(x),$$

$$T_{34}(x) = \max\{0, 1 - |x - 3|\}, \quad I_{34}(x) = 0.1 \text{ on } [2, 4], \quad F_{34}(x) = 1 - T_{34}(x).$$

Given planned releases $x = (x_1, x_2, x_3)$, neutrosophic node times $\tilde{T}_i(x, \tilde{n})$ and completion time

$$\tilde{T}(x, \tilde{n}) = \tilde{T}_3(x, \tilde{n}) \oplus \tilde{n}_{34}$$

are obtained by the componentwise extension of + and max (sup–min for T and its duals for I, F). Management sets a target deadline $T_0 = 16$ days and requires

$$\text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq T_0\} \geq 0.80,$$

while minimizing a neutrosophic expected cost $\mathbb{E}_N[\tilde{C}(x, \tilde{n})]$ (e.g., overtime penalties if the project extends). The scheduler chooses start times x^* satisfying the credibility constraint and achieving the smallest $\mathbb{E}_N[\tilde{C}(x, \tilde{n})]$, explicitly accounting for truth, indeterminacy, and falsity in each activity’s duration.

Theorem 3.3 (NPS generalizes fuzzy project scheduling). *Assume each fuzzy duration $\tilde{n}_{ij}^F$ is given by a membership $\mu_{ij} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$. Embed fuzzy numbers into neutrosophic numbers via*

$$\iota_{F \rightarrow N} : \tilde{n}_{ij}^F \mapsto \tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad T_{ij} = \mu_{ij}, \quad I_{ij} \equiv 0, \quad F_{ij} = 1 - \mu_{ij}.$$

Let Cr_N and $\mathbb{E}_N$ reduce on such embeddings to the fuzzy credibility Cr and the fuzzy expectation $\mathbb{E}$ computed from T (i.e. for any fuzzy event A , $\text{Cr}_N(A^N) = \text{Cr}(A)$ and $\mathbb{E}_N[\tilde{Z}^N] = \mathbb{E}[\tilde{Z}]$). Then, under the componentwise extension principle, the feasible set and optimal value of NPS coincide with those of the corresponding fuzzy project scheduling problem.

Proof. By construction, the T –components of $\tilde{T}_i(x, \tilde{n})$ and $\tilde{T}(x, \tilde{n})$ are exactly the fuzzy start/finish-time membership functions obtained by applying Zadeh’s extension principle to $(+, \max)$ with inputs μ_{ij} . Since $I \equiv 0$ and $F = 1 - T$ are preserved by the same principle on these embeddings, every neutrosophic event of the form $\{\tilde{Z} \leq z\}$ produces T –marginals identical to the fuzzy event membership for $\{Z \leq z\}$. By the reduction property of Cr_N and $\mathbb{E}_N$, the chance and expectation constraints in **(N-cred)** and **(N-exp)** coincide with their fuzzy counterparts. Therefore the feasible region and objective values are the same in both models, proving that NPS strictly generalizes fuzzy project scheduling. $\square$

3.2 Plithogenic Project Scheduling

Plithogenic Project Scheduling represents activity durations via attribute-driven memberships and contradiction degrees, aggregating to fuzzy or neutrosophic times for optimization.

Definition 3.4 (Plithogenic Project Scheduling (PPS)). Retain the network $G = (V, A)$. For each activity $(i, j) \in A$, let its duration be described by a plithogenic description

$$\tilde{p}_{ij} = (pdf_{ij}, pCF_{ij}),$$

where $pdf_{ij} : \mathbb{R}_{\geq 0} \times Pv \rightarrow [0, 1]^s$ gives the degree(s) of appurtenance at duration x for each attribute value $a \in Pv$, and $pCF_{ij} : Pv \times Pv \rightarrow [0, 1]^t$ is the degree of contradiction. Let $\mathcal{A}_{pCF}$ be a plithogenic aggregation operator that, for each x , merges $\{pdf_{ij}(x, a) : a \in Pv\}$ into either:

- (i) an *effective fuzzy membership* $\mu_{ij}^{\text{eff}}(x) \in [0, 1]$, or
 - (ii) an *effective neutrosophic triple* $(T_{ij}^{\text{eff}}, I_{ij}^{\text{eff}}, F_{ij}^{\text{eff}})(x) \in [0, 1]^3$,
- consistently across x and (i, j) .

- **Fuzzy-aggregated PPS:** If (i) is chosen, define fuzzy start/finish membership by the extension principle on $(+, \max)$ using μ_{ij}^{eff} , and solve the fuzzy chance/expectation model with credibility Cr and expectation $\mathbb{E}$.
- **Neutrosophic-aggregated PPS:** If (ii) is chosen, define neutrosophic start/finish numbers via componentwise extension on $(+, \max)$ using $(T_{ij}^{\text{eff}}, I_{ij}^{\text{eff}}, F_{ij}^{\text{eff}})$, and solve the NPS model with Cr_N and $\mathbb{E}_N$.

In both cases, call the resulting optimization problem the *PPS problem* (with the chosen aggregation mode).

Example 3.5 (Plithogenic Project Scheduling — Coastal Bridge Repair). A coastal bridge repair project faces attribute-dependent durations due to *weather*. Let the attribute be $v = \text{weather}$ with value domain $Pv = \{\text{dry}, \text{stormy}\}$. For the critical activity “deck resurfacing” (i, j) (hours), the plithogenic duration is

$$\tilde{n}_{ij} = (pdf_{ij}(\cdot; \text{dry}), pdf_{ij}(\cdot; \text{stormy})),$$

with (componentwise) degrees of appurtenance

$$pdf_{ij}(x; \text{dry}) = \max\{0, 1 - |x - 10|/2\}, \quad pdf_{ij}(x; \text{stormy}) = \max\{0, 1 - |x - 14|/3\}.$$

Contradiction between weather values is encoded by

$$pCF(\text{dry}, \text{stormy}) = pCF(\text{stormy}, \text{dry}) = 0.8, \quad pCF(a, a) = 0.$$

With t -norm $T(x, y) = \min\{x, y\}$ and t -conorm $S(x, y) = \max\{x, y\}$, the plithogenic aggregator for combining two steps with attribute values $a, b \in Pv$ uses

$$A_c(x, y) = (1 - c)T(x, y) + cS(x, y), \quad c = pCF(a, b).$$

Start/finish times and the project completion time are propagated by the plithogenic lifts of $+$ and $\max$ using A_c , i.e. for U with carrier a and V with carrier b ,

$$\mu_{U \circ V}(z) = \sup_{\circ(u, v)=z} A_{pCF(a, b)}(\mu_U(u), \mu_V(v)), \quad \circ \in \{+, \max\}.$$

The planner enforces a deadline T_0 via a plithogenic credibility $\text{Cr}_p\{\tilde{T} \leq T_0\} \geq 0.85$ and minimizes a plithogenic expected cost $\mathbb{E}_p[\tilde{C}]$ that includes weather-dependent crew premiums. Because pCF is high, disagreement between “dry” and “stormy” pushes aggregation toward the t -conorm side, widening the effective duration and influencing the optimal start times and resource allocation under the bridge closure window.

Theorem 3.6 (PPS generalizes FPS and NPS). (a) (FPS as a special case) Let $Pv = \{a^*\}$ be a singleton, $s = 1$, and define $pdf_{ij}(x, a^*) = \mu_{ij}(x)$ with any pCF_{ij} . Choose $\mathcal{A}_{pCF}$ to return $\mu_{ij}^{\text{eff}}(x) = pdf_{ij}(x, a^*)$. Then the fuzzy-aggregated PPS coincides with the fuzzy project scheduling problem.

(b) (NPS as a special case) Let $Pv = \{T, I, F\}$ and $s = 1$, and set

$$pdf_{ij}(x, T) = T_{ij}(x), \quad pdf_{ij}(x, I) = I_{ij}(x), \quad pdf_{ij}(x, F) = F_{ij}(x),$$

with any contradiction map pCF_{ij} satisfying $pCF_{ij}(T, F) = 1$ and $pCF_{ij}(T, T) = pCF_{ij}(I, I) = pCF_{ij}(F, F) = 0$. Choose $\mathcal{A}_{pCF}$ to return the triple $(T_{ij}^{\text{eff}}, I_{ij}^{\text{eff}}, F_{ij}^{\text{eff}}) = (pdf_{ij}(\cdot, T), pdf_{ij}(\cdot, I), pdf_{ij}(\cdot, F))$ (i.e., identity aggregation). Then the neutrosophic-aggregated PPS coincides with NPS.

Proof. (a) With a single attribute value, the aggregation $\mathcal{A}_{pCF}$ yields $\mu_{ij}^{\text{eff}} = \mu_{ij}$ pointwise, so the fuzzy forward/backward propagation via the extension principle and the resulting credibility/expectation constraints are exactly those of the fuzzy model. Hence $\text{PPS} \equiv \text{FPS}$.

(b) With $Pv = \{T, I, F\}$ and identity aggregation, the effective plithogenic description recovers the given neutrosophic triples (T_{ij}, I_{ij}, F_{ij}) . Therefore the componentwise extension of $(+, \max)$ and the neutrosophic chance/expectation constraints coincide with those of NPS by definition. Hence $\text{PPS} \equiv \text{NPS}$. $\square$

3.3 Fuzzy Project MultiScheduling

The Fuzzy Project MultiScheduling model treats schedules as multisets, combines candidates via multi-operations, and selects a representative minimizing fuzzy cost.

Definition 3.7 (Fuzzy Project MultiScheduling (FPMS)). Let $G = (V, A)$ be a finite DAG with source 1 and sink $n+1$. For each activity $(i, j) \in A$, the duration is a fuzzy number $\tilde{n}_{ij}$ (arbitrary normal, convex fuzzy number on $\mathbb{R}_{\geq 0}$). A (single) *schedule* is a release/start vector $x = (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n$. For any x , define the fuzzy node times $\tilde{T}_i(x, \tilde{n})$ and the fuzzy project completion time

$$\tilde{T}(x, \tilde{n}) = \max_{(k, n+1) \in A} (\tilde{T}_k(x, \tilde{n}) + \tilde{n}_{k, n+1})$$

via Zadeh's extension principle applied to $(u, v) \mapsto u+v$ and $(u, v) \mapsto \max\{u, v\}$. Let $\tilde{C}(x, \tilde{n})$ be a fuzzy cost (e.g. time–cost accrual under fuzzy times). Fix a fuzzy expectation $\mathbb{E}[\cdot]$ and a credibility measure $\text{Cr}\{\cdot\}$.

MultiStructure of schedules. Let $H := \mathbb{R}_{\geq 0}^n$ be the set of all (single) schedules. For each arity $m \in \mathbb{N}$, fix a finite family of *combiner maps*

$$\mathcal{A}_m = \{A_{m,1}, \dots, A_{m,q_m}\}, \quad A_{m,\ell} : H^m \rightarrow H,$$

and define the multi-operation $\#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ by

$$\#^{(m)}(x^{(1)}, \dots, x^{(m)}) = \{A_{m,1}(x^{(1)}, \dots, x^{(m)}), \dots, A_{m,q_m}(x^{(1)}, \dots, x^{(m)})\},$$

counting multiplicity when outputs coincide. Typical choices include

$$A_{m,\ell}(x^{(1)}, \dots, x^{(m)}) \in \left\{ \min_{\text{coord}} \max_{\text{coord}} \sum_{r=1}^m \lambda_r^{(\ell)} x^{(r)} \right\}, \quad \lambda^{(\ell)} \in \Delta_m \cap \mathbb{Q}^m,$$

which keep $A_{m,\ell}(H^m) \subseteq H$. Then

$$\mathcal{MS} := (H, \{\#^{(m)}\}_{m \geq 1})$$

is a *MultiStructure* in the sense $H^m \xrightarrow{\#^{(m)}} \mathcal{M}(H)$.

Reachable multischedules and selection. Given a finite *seed set* $S_0 \subset H$, let $\text{Cl}_{\mathcal{MS}}(S_0)$ be the smallest subset of H that contains S_0 and is closed under: (i) extraction of elements from any multiset in the image of $\#^{(m)}$, and (ii) repeated application of all $\#^{(m)}$ on tuples drawn from the current set. A *multischedule* is any finite multiset $M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))$. A *selector* $\text{Sel} : \mathcal{M}(H) \rightarrow H$ picks a representative schedule, e.g.

$$\text{Sel}(M) \in \arg \min_{x \in \text{supp}(M)} \{ \mathbb{E}[\tilde{C}(x, \tilde{n})] : \text{Cr}\{\tilde{T}(x, \tilde{n}) \leq T_0\} \geq \beta \}$$

(with deterministic tie-breaking).

Optimization model. The FPMS problem (EVM form) is

$$\min_{M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))} \mathbb{E}[\tilde{C}(\text{Sel}(M), \tilde{n})] \quad \text{s.t.} \quad \mathbb{E}[\tilde{T}(\text{Sel}(M), \tilde{n})] \leq T_0.$$

Equivalently (credibility form),

$$\min_{M, C_0} C_0 \quad \text{s.t.} \quad \text{Cr}\{\tilde{T}(\text{Sel}(M), \tilde{n}) \leq T_0\} \geq \beta, \quad \text{Cr}\{\tilde{C}(\text{Sel}(M), \tilde{n}) \leq C_0\} \geq \alpha.$$

Example 3.8 (Fuzzy Project MultiScheduling — Software Feature Rollout). A software team releases a feature via three serial activities on a DAG $G = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (3, 4)\}$: coding (1, 2), testing (2, 3), and deployment (3, 4). Activity durations (days) are triangular fuzzy numbers

$$\tilde{n}_{12} = \text{TFN}(4, 5, 7), \quad \tilde{n}_{23} = \text{TFN}(2, 3, 4), \quad \tilde{n}_{34} = \text{TFN}(1, 2, 3).$$

Two candidate (single) schedules on $H = \mathbb{R}_{\geq 0}^3$ are $x^{(A)} = (0, 0, 0)$ (earliest starts) and $x^{(B)} = (0, 1, 1)$ (smoothed). The multi-operation of arity $m = 2$ produces a multischedule $M = \{x^{(A)}, x^{(B)}, \frac{1}{2}x^{(A)} + \frac{1}{2}x^{(B)} = (0, 0.5, 0.5)\}$. For a serial chain, the fuzzy completion time with $x^{(A)}$ is

$$\tilde{T}(x^{(A)}, \tilde{n}) = \tilde{n}_{12} \oplus \tilde{n}_{23} \oplus \tilde{n}_{34} = \text{TFN}(7, 10, 14),$$

whose centroid $\frac{7+10+14}{3} = 10.\bar{3}$ satisfies the target $\mathbb{E}[\tilde{T}] \leq T_0 = 12$. Let the fuzzy project cost be $\tilde{C} = \tilde{T} + \tilde{P}$, where $\tilde{P}$ penalizes burst staffing implied by tight starts. The selector Sel chooses, among the support of M ,

$$\text{Sel}(M) \in \arg \min \{ \mathbb{E}[\tilde{C}(x, \tilde{n})] : \mathbb{E}[\tilde{T}(x, \tilde{n})] \leq 12 \}.$$

Because $x^{(B)}$ and the blend $(0, 0.5, 0.5)$ reduce $\tilde{P}$ without worsening $\tilde{T}$ for these durations, $\text{Sel}(M)$ returns a smoothed option (typically the blend), illustrating how multi-scheduling aggregates candidate plans and picks a representative under fuzzy expectation and deadline.

Example 3.9 (Fuzzy Project MultiScheduling — University Data-Center Migration). A university IT office must migrate services from an on-premise data center to a new hybrid cloud. The migration project is modeled by a serial DAG

$$G = (V, A), \quad V = \{1, 2, 3, 4, 5\}, \quad A = \{(1, 2), (2, 3), (3, 4), (4, 5)\},$$

with activities: (1,2) “network backbone reconfiguration,” (2,3) “VM image export,” (3,4) “database switchover,” (4,5) “service validation.” Because each activity depends on staff availability and vendor coordination, durations are fuzzy.

Let the activity durations (in hours) be triangular fuzzy numbers:

$$\tilde{n}_{1,2} = \text{TFN}(3, 4, 6), \quad \tilde{n}_{2,3} = \text{TFN}(5, 6, 8), \quad \tilde{n}_{3,4} = \text{TFN}(2, 3, 4), \quad \tilde{n}_{4,5} = \text{TFN}(1, 2, 3).$$

Two realistic single schedules on $H = \mathbb{R}_{\geq 0}^4$ are prepared:

$$x^{(D)} = (0, 0, 0, 0) \quad (\text{“dense”}: \text{do all as early as possible}),$$

$$x^{(O)} = (0, 1, 2, 3) \quad (\text{“off-peak”}: \text{shift DB and validation to off-hours}).$$

Define, for arity $m = 2$, the multi-operation

$$\#^{(2)}(x^{(1)}, x^{(2)}) = \left\{ \min_{\text{coord}}(x^{(1)}, x^{(2)}), \max_{\text{coord}}(x^{(1)}, x^{(2)}), \frac{1}{2}(x^{(1)} + x^{(2)}) \right\},$$

so that from $(x^{(D)}, x^{(O)})$ we obtain the multischedule

$$M = \{(0, 0, 0, 0), (0, 1, 2, 3), (0, 0.5, 1, 1.5)\}.$$

For a serial chain, the fuzzy completion time for a schedule x is

$$\tilde{T}(x, \tilde{n}) = \tilde{n}_{1,2} \oplus \tilde{n}_{2,3} \oplus \tilde{n}_{3,4} \oplus \tilde{n}_{4,5},$$

so, for $x^{(D)}$, by summing TFNs componentwise,

$$\tilde{T}(x^{(D)}, \tilde{n}) = \text{TFN}(3+5+2+1, 4+6+3+2, 6+8+4+3) = \text{TFN}(11, 15, 21).$$

Its centroid is $(11 + 15 + 21)/3 = 15.67$ hours. Suppose the university wants

$$\mathbb{E}[\tilde{T}(x, \tilde{n})] \leq T_0 = 16 \quad \text{and} \quad \text{Cr}\{\tilde{T}(x, \tilde{n}) \leq 18\} \geq 0.85.$$

Let the fuzzy cost be $\tilde{C}(x, \tilde{n}) = \tilde{T}(x, \tilde{n}) + \tilde{O}(x)$, where $\tilde{O}(x)$ is a fuzzy “overtime or off-peak premium” that is higher for schedules starting at 0 and lower for off-peak/shifted schedules.

The selector

$$\text{Sel}(M) \in \arg \min_{x \in \text{supp}(M)} \left\{ \mathbb{E}[\tilde{C}(x, \tilde{n})] : \text{Cr}\{\tilde{T}(x, \tilde{n}) \leq 18\} \geq 0.85 \right\}$$

will typically pick the blended schedule $(0, 0.5, 1, 1.5)$, because it satisfies the time requirement (the chain is still about 16 hours in expectation) while reducing the fuzzy off-peak premium compared with the dense schedule. This shows how fuzzy project multischeduling keeps several candidate plans and then picks the best under fuzzy expectation and credibility.

Proposition 3.10 (Schedule space is a MultiStructure). *The pair $\mathcal{MS} = (H, \{\#^{(m)}\}_{m \geq 1})$ defined above is a MultiStructure: for every m and every $(x^{(1)}, \dots, x^{(m)}) \in H^m$, $\#^{(m)}(x^{(1)}, \dots, x^{(m)})$ is a finite multiset in $\mathcal{M}(H)$.*

Proof. Each $\mathcal{A}_m$ is finite, hence the image $\{A_{m,\ell}(x^{(1)}, \dots, x^{(m)}) : \ell = 1, \dots, q_m\}$ is a finite subset of H ; counting duplicates yields an element of $\mathcal{M}(H)$ by Definition of finite multiset. Thus $\#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ is well-defined for all m , which is precisely the MultiStructure axiom. $\square$

Theorem 3.11 (FPMS generalizes fuzzy project scheduling). *Let $S_0 = H$ and take the degenerate MultiStructure given by*

$$\mathcal{A}_1 = \{\text{Id}\}, \quad \mathcal{A}_m = \emptyset \quad (m \geq 2), \quad \text{so} \quad \#^{(1)}(x) = \{x\}, \quad \#^{(m)} \text{ undefined}.$$

Let Sel be the identity on singletons: $\text{Sel}(\{x\}) = x$. Then the feasible set of FPMS equals the set of (single) schedules H , and FPMS reduces exactly to the classical fuzzy project scheduling problem (in either EVM or credibility form).

Proof. With $S_0 = H$ and only $\#^{(1)}(x) = \{x\}$ available, closure adds nothing: $\text{Cl}_{\mathcal{MS}}(S_0) = H$. Any admissible multischedule M must be a finite multiset of elements of H generated solely by identity, hence every singleton $\{x\}$ with $x \in H$ is feasible and $\text{Sel}(\{x\}) = x$. The FPMS objective and constraints then read

$$\min_{x \in H} \mathbb{E}[\tilde{C}(x, \tilde{n})] \quad \text{s.t.} \quad \mathbb{E}[\tilde{T}(x, \tilde{n})] \leq T_0,$$

or, in credibility form,

$$\min_{x, C_0} C_0 \quad \text{s.t.} \quad \text{Cr}\{\tilde{T}(x, \tilde{n}) \leq T_0\} \geq \beta, \quad \text{Cr}\{\tilde{C}(x, \tilde{n}) \leq C_0\} \geq \alpha,$$

which are exactly the standard fuzzy project scheduling formulations. Hence FPMS strictly contains (and therefore generalizes) fuzzy project scheduling. $\square$

3.4 Neutrosophic Project MultiScheduling

The Neutrosophic Project MultiScheduling framework aggregates multisets of schedules under truth–indeterminacy–falsity durations and selects representatives optimizing expectation and credibility criteria.

Definition 3.12 (Neutrosophic Project MultiScheduling (NPMS)). Let the project be a finite DAG $G = (V, A)$ with source 1 and sink $n+1$. For each activity $(i, j) \in A$, the duration is a *single-valued neutrosophic number* (SVNN)

$$\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad T_{ij}, I_{ij}, F_{ij} : \mathbb{R}_{\geq 0} \rightarrow [0, 1],$$

interpreting, for $x \geq 0$, the degrees that “ $d_{ij} = x$ ” is *true* ($T_{ij}(x)$), *indeterminate* ($I_{ij}(x)$), and *false* ($F_{ij}(x)$). A (single) schedule is a vector $x = (x_1, \dots, x_n) \in H := \mathbb{R}_{\geq 0}^n$.

Neutrosophic time propagation. Write E for Zadeh’s extension principle applied *componentwise* to the scalar maps $(u, v) \mapsto u+v$ and $(u, v) \mapsto \max\{u, v\}$: for neutrosophic numbers $U = (T_U, I_U, F_U)$ and $V = (T_V, I_V, F_V)$ and an operation $\circ \in \{+, \max\}$,

$$\begin{aligned} T_{U \circ V}(z) &= \sup_{\circ(u,v)=z} \min\{T_U(u), T_V(v)\}, \\ I_{U \circ V}(z) &= \sup_{\circ(u,v)=z} \min\{I_U(u), I_V(v)\}, \\ F_{U \circ V}(z) &= \inf_{\circ(u,v)=z} \max\{F_U(u), F_V(v)\}. \end{aligned}$$

(Any continuous t –norm/ s –norm pair yields an equivalent formulation.) Define neutrosophic node times recursively by

$$\tilde{T}_1(x, \tilde{n}) = \tilde{x}_1, \quad \tilde{T}_i(x, \tilde{n}) \succeq \tilde{x}_i, \quad \tilde{T}_i(x, \tilde{n}) \succeq \bigvee_{(k,i) \in A} (\tilde{T}_k(x, \tilde{n}) \oplus \tilde{n}_{ki}),$$

where $\oplus$ (resp. $\vee$) denotes E–lift of $+$ (resp. $\max$), and $\succeq$ is the product order on SVNNs. The neutrosophic completion time is

$$\tilde{T}(x, \tilde{n}) = \bigvee_{(k,n+1) \in A} (\tilde{T}_k(x, \tilde{n}) \oplus \tilde{n}_{k,n+1}).$$

MultiStructure of schedules. Let $\mathcal{M}(H)$ denote the set of *finite multisets* on H . For each arity $m \geq 1$ choose a finite family of *combiner maps*

$$\mathcal{A}_m = \{A_{m,1}, \dots, A_{m,q_m}\}, \quad A_{m,\ell} : H^m \rightarrow H,$$

and define the multi-operation $\#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ by

$$\#^{(m)}(x^{(1)}, \dots, x^{(m)}) = \left\{ A_{m,1}(\vec{x}), \dots, A_{m,q_m}(\vec{x}) \right\} \in \mathcal{M}(H), \quad \vec{x} = (x^{(1)}, \dots, x^{(m)}).$$

Typical choices include coordinatewise min / max and rational convex combinations $A_{m,\ell}(\vec{x}) = \sum_{r=1}^m \lambda_r^{(\ell)} x^{(r)}$ with $\lambda^{(\ell)} \in \Delta_m \cap \mathbb{Q}^m$. Then

$$\mathcal{MS} = (H, \{\#^{(m)}\}_{m \geq 1})$$

is a *MultiStructure*.

Reachable multischedules, selection, and cost. Given a finite seed set $S_0 \subset H$, let $\text{Cl}_{\mathcal{MS}}(S_0)$ be the smallest subset of H that contains S_0 and is closed under taking elements of any $\#^{(m)}(\cdot)$ -image and under repeated applications of all $\#^{(m)}$. A *multischedule* is any $M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))$. Let $\text{Sel} : \mathcal{M}(H) \rightarrow H$ select a representative schedule (e.g. by neutrosophic optimization below). Let the (possibly neutrosophic) project cost be an SVN $\tilde{C}(x, \tilde{n})$ obtained from $\tilde{T}(\cdot)$ by E-lifting a scalar cost model (e.g. time-cost accrual).

Neutrosophic optimization. Let Cr_N be a neutrosophic credibility and $\mathbb{E}_N[\cdot]$ a neutrosophic expectation. Two canonical NPMS formulations are:

$$\begin{aligned} \text{(NPMS-EVM)} \quad & \min_{M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))} \mathbb{E}_N[\tilde{C}(\text{Sel}(M), \tilde{n})] \\ & \text{s.t. } \mathbb{E}_N[\tilde{T}(\text{Sel}(M), \tilde{n})] \leq T_0; \\ \text{(NPMS-Cred)} \quad & \min_{M, C_0} C_0 \\ & \text{s.t. } \text{Cr}_N\{\tilde{T}(\text{Sel}(M), \tilde{n}) \leq T_0\} \geq \beta, \\ & \text{Cr}_N\{\tilde{C}(\text{Sel}(M), \tilde{n}) \leq C_0\} \geq \alpha. \end{aligned}$$

Example 3.13 (Neutrosophic Project MultiScheduling — City Vaccination Pods). A city opens pop-up vaccination pods through three serial activities on $G = (V, A)$: site preparation (1, 2), cold-chain delivery (2, 3), and station setup (3, 4). Each duration is a single-valued neutrosophic number $\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij})$ (hours). For delivery,

$$T_{23}(x) = \max\{0, 1 - |x - 6|/2\}, \quad I_{23}(x) = 0.25 \text{ on } [5, 7], 0 \text{ else}, \quad F_{23}(x) = 1 - T_{23}(x).$$

Analogous triples are specified for (1, 2) centered at 4 and for (3, 4) centered at 3. Two seed schedules are $x^{(F)} = (0, 0, 0)$ (fast) and $x^{(B)} = (0, 0.5, 1)$ (with buffers). The multi-operation yields the multi-schedule $M = \{x^{(F)}, x^{(B)}, \frac{1}{2}(x^{(F)} + x^{(B)})\}$. Neutrosophic node/finish times use the componentwise extension of $+$ and $\max$ (sup-min on T, I and the standard dual on F). The health department imposes

$$\text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq T_0 = 14\} \geq 0.85,$$

and minimizes $\mathbb{E}_N[\tilde{C}(x, \tilde{n})]$ (overtime and spoilage risk). Evaluating the support of M , the buffered plan (or its blend) typically raises the credibility of meeting T_0 while controlling expected cost, so $\text{Sel}(M)$ returns the buffered option, exemplifying neutrosophic multi-scheduling under truth-indeterminacy-falsity.

Example 3.14 (Neutrosophic Project MultiScheduling — Emergency Shelter Deployment). A municipal disaster office needs to deploy an emergency shelter after a typhoon. The project is a DAG $G = (V, A)$ with

$$V = \{1, 2, 3, 4\}, \quad A = \{(1, 2), (1, 3), (2, 4), (3, 4)\},$$

representing two parallel activity lines: (1, 2) “site preparation,” (1, 3) “supplies reception,” and finally (2, 4) and (3, 4) “setup and opening.” Because information after a disaster is incomplete, each duration is a single-valued neutrosophic number (SVNN).

Durations (hours) are specified as follows. For site preparation (1, 2):

$$T_{12}(x) = \max\{0, 1 - |x - 4|/1.5\}, \quad I_{12}(x) = 0.25 \text{ on } [3, 5], \quad F_{12}(x) = 1 - T_{12}(x).$$

For supplies reception (1, 3):

$$T_{13}(x) = \max\{0, 1 - |x - 5|/2\}, \quad I_{13}(x) = 0.30 \text{ on } [4, 6], \quad F_{13}(x) = 1 - T_{13}(x).$$

For the two final activities (2, 4) and (3, 4) we take SVNNs centered at 3 hours with 0.2 indeterminacy.

Two seed schedules on $H = \mathbb{R}_{\geq 0}^3$ (for nodes 2, 3, 4) are

$$x^{(E)} = (0, 0, 0) \quad (\text{emergency: open as soon as each branch is done}),$$

$$x^{(S)} = (0, 1, 1) \quad (\text{staggered: allow 1h buffer for late supplies}).$$

Define the neutrosophic multi-operation of arity 2 by

$$\#^{(2)}(x^{(1)}, x^{(2)}) = \{\min_{\text{coord}}(x^{(1)}, x^{(2)}), \max_{\text{coord}}(x^{(1)}, x^{(2)}), \frac{1}{2}(x^{(1)} + x^{(2)})\},$$

so that we obtain the multischedule

$$M = \{(0, 0, 0), (0, 1, 1), (0, 0.5, 0.5)\}.$$

Neutrosophic start/finish times are computed by the componentwise extension of $+$ and $\max$, so the neutrosophic completion time for (0, 0, 0) is

$$\tilde{T}(x^{(E)}, \tilde{n}) = (\tilde{T}_2(x^{(E)}, \tilde{n}) \oplus \tilde{n}_{24}) \vee (\tilde{T}_3(x^{(E)}, \tilde{n}) \oplus \tilde{n}_{34}),$$

where $\oplus$ and $\vee$ are the neutrosophic lifts. The disaster office requires

$$\text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq 10\} \geq 0.80 \quad \text{and} \quad \mathbb{E}_N[\tilde{C}(x, \tilde{n})] \text{ minimal},$$

where $\tilde{C}$ includes neutrosophic penalties for staff overtime (which increases falsity if staff is over-stretched) and for late opening (which increases I).

Define the selector

$$\text{Sel}(M) \in \arg \min_{x \in \text{supp}(M)} \{\mathbb{E}_N[\tilde{C}(x, \tilde{n})] : \text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq 10\} \geq 0.80\}.$$

The pure emergency schedule (0, 0, 0) opens fastest but may have higher indeterminacy in the final setup because supplies can still be in flux; the staggered schedule (0, 1, 1) reduces indeterminacy in the last step. In many parameter settings the selector chooses the blended (0, 0.5, 0.5), which balances truth (feasibility), indeterminacy (delays still possible), and falsity (overtime not too high). This illustrates neutrosophic project multischeduling in an emergency context.

Proposition 3.15 (The schedule space is a MultiStructure). *For every $m \geq 1$ and $\vec{x} \in H^m$, $\#^{(m)}(\vec{x}) \in \mathcal{M}(H)$ is finite. Hence $\mathcal{MS} = (H, \{\#^{(m)}\}_{m \geq 1})$ is a MultiStructure.*

Proof. Each $\mathcal{A}_m$ is finite; therefore $\{A_{m,\ell}(\vec{x})\}_{\ell=1}^{q_m}$ is a finite subset of H . Counting duplicates gives a finite multiset in $\mathcal{M}(H)$, which is precisely the MultiStructure requirement. $\square$

Theorem 3.16 (NPMS generalizes Neutrosophic Project Scheduling). *Let $S_0 = H$, take the degenerate MultiStructure with $\mathcal{A}_1 = \{\text{Id}\}$ and $\mathcal{A}_m = \emptyset$ for $m \geq 2$, hence $\#^{(1)}(x) = \{x\}$, and define $\text{Sel}(\{x\}) = x$. Then NPMS–EVM (resp. NPMS–Cred) reduces exactly to the neutrosophic project scheduling model (EVM, resp. credibility form).*

Proof. With only $\#^{(1)}$ as identity, $\text{Cl}_{\mathcal{MS}}(S_0) = H$ and every feasible M can be chosen as a singleton $\{x\}$. Because $\text{Sel}(\{x\}) = x$, the NPMS objectives/constraints become

$$\min_{x \in H} \mathbb{E}_N[\tilde{C}(x, \tilde{n})] \quad \text{s.t.} \quad \mathbb{E}_N[\tilde{T}(x, \tilde{n})] \leq T_0,$$

or, respectively,

$$\min_{x, C_0} C_0 \quad \text{s.t.} \quad \text{Cr}_N\{\tilde{T}(x, \tilde{n}) \leq T_0\} \geq \beta, \quad \text{Cr}_N\{\tilde{C}(x, \tilde{n}) \leq C_0\} \geq \alpha,$$

which are precisely the standard neutrosophic scheduling formulations. $\square$

Theorem 3.17 (NPMS generalizes Fuzzy Project MultiScheduling). *Suppose each fuzzy duration $\tilde{n}_{ij}^F$ has membership $\mu_{ij} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$. Embed fuzzy numbers into SVNNS via*

$$\iota_{F \rightarrow N} : \tilde{n}_{ij}^F \mapsto \tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad T_{ij} = \mu_{ij}, \quad I_{ij} \equiv 0, \quad F_{ij} = 1 - \mu_{ij}.$$

Assume Cr_N and $\mathbb{E}_N$ reduce on such embeddings to the fuzzy credibility Cr and fuzzy expectation $\mathbb{E}$: for any fuzzy event/value Z , $\text{Cr}_N(Z^N) = \text{Cr}(Z)$ and $\mathbb{E}_N[\tilde{Z}^N] = \mathbb{E}[\tilde{Z}]$. Let the MultiStructure $\mathcal{MS}$ be the same one used in the FPMS model and keep the same selector Sel . Then NPMS–EVM/NPMS–Cred coincide with FPMS (EVM/credibility).

Proof. Under $\iota_{F \rightarrow N}$, the T -components of all constructed SVNNS (node times, completion time, cost) are exactly the fuzzy memberships obtained by Zadeh's extension principle, because our componentwise lifting uses the same sup–min (for T) as in the fuzzy case. Since $I \equiv 0$ and $F = 1 - T$ are preserved by the componentwise lift, the T -marginal of any neutrosophic event $\{\tilde{Z} \leq z\}$ equals the fuzzy membership of $\{Z \leq z\}$. By reduction, Cr_N and $\mathbb{E}_N$ return Cr and $\mathbb{E}$ on these embeddings. Because $\mathcal{MS}$ and Sel are unchanged, the feasible multischedules and the selected representatives coincide with those in FPMS, and the objectives/constraints evaluate to the same real values. Hence the NPMS formulations are identical to FPMS. $\square$

3.5 Plithogenic Project MultiScheduling

The Plithogenic Project MultiScheduling model integrates memberships and contradiction functions, forms multisets, and selects representatives optimizing expectation and credibility criteria.

Definition 3.18 (Plithogenic Project MultiScheduling (PPMS)). *Let the project network be a finite DAG $G = (V, A)$ with source 1 and sink $n+1$. A (single) schedule is a vector $x = (x_1, \dots, x_n) \in H := \mathbb{R}_{\geq 0}^n$. Fix a plithogenic attribute v with value domain Pv and a degree of contradiction $pCF : Pv \times Pv \rightarrow [0, 1]$ (reflexive and symmetric). For each activity $(i, j) \in A$ we are given a *plithogenic duration**

$$\tilde{n}_{ij} = (pdf_{ij}(\cdot; a_{ij}))_{a_{ij} \in Pv}, \quad pdf_{ij}(\cdot; a) : \mathbb{R}_{\geq 0} \rightarrow [0, 1]^s,$$

where $s \in \mathbb{N}$ is the (fixed) appurtenance dimension and $a_{ij} \in Pv$ is the activity's attribute value. For $z \geq 0$ and component $r = 1, \dots, s$, write $pdf_{ij}^{(r)}(z; a)$ for the r th coordinate.

Plithogenic lift of + and max. Let $T, S : [0, 1]^2 \rightarrow [0, 1]$ be a fixed t -norm/ t -conorm (e.g. $T(x, y) = \min\{x, y\}$, $S(x, y) = \max\{x, y\}$). Given two plithogenic numbers U with carrier attribute $a \in Pv$ and V with carrier attribute $b \in Pv$, set $c := pCF(a, b) \in [0, 1]$ and define the *plithogenic aggregator*

$$A_c(x, y) = (1 - c)T(x, y) + cS(x, y), \quad x, y \in [0, 1].$$

For a binary operation $\circ \in \{+, \max\}$, the componentwise plithogenic extension E_p is

$$\mu_{U \circ V}^{(r)}(z) = \sup_{\circ(u, v)=z} A_c(\mu_U^{(r)}(u), \mu_V^{(r)}(v)), \quad r = 1, \dots, s,$$

with carrier attribute propagated by a chosen rule, e.g. $a(U \circ V) \in \{a, b\}$ (any fixed policy suffices for the results below).

Node times and completion time. Let $\tilde{x}_i$ be the degenerate plithogenic number concentrated at x_i (all coordinates equal to 1 at x_i and 0 elsewhere). Define $\tilde{T}_i(x, \tilde{n})$ as the *least* family of plithogenic numbers (w.r.t. componentwise order on $[0, 1]^s$) satisfying

$$\tilde{T}_1(x, \tilde{n}) = \tilde{x}_1, \quad \tilde{T}_i(x, \tilde{n}) \succeq \tilde{x}_i, \quad \tilde{T}_i(x, \tilde{n}) \succeq \bigvee_p \left\{ \tilde{T}_k(x, \tilde{n}) \oplus_p \tilde{n}_{ki} : (k, i) \in A \right\},$$

where $\oplus_p$ and $\bigvee_p$ are the E_p -lifts of $+$ and $\max$, respectively, and $\succeq$ denotes the product order on $[0, 1]^s$ -valued membership functions. The plithogenic completion time is

$$\tilde{T}(x, \tilde{n}) = \bigvee_p \left\{ \tilde{T}_k(x, \tilde{n}) \oplus_p \tilde{n}_{k, n+1} : (k, n+1) \in A \right\}.$$

Let $\tilde{C}(x, \tilde{n})$ be a plithogenic cost obtained by E_p -lifting a scalar cost model (e.g. time-cost accrual).

MultiStructure of schedules. For each arity $m \geq 1$, choose a finite family of combiner maps

$$\mathcal{A}_m = \{A_{m,1}, \dots, A_{m,q_m}\}, \quad A_{m,\ell} : H^m \rightarrow H,$$

and define the multi-operation $\#^{(m)} : H^m \rightarrow \mathcal{M}(H)$ by

$$\#^{(m)}(x^{(1)}, \dots, x^{(m)}) = \{A_{m,1}(\vec{x}), \dots, A_{m,q_m}(\vec{x})\}, \quad \vec{x} = (x^{(1)}, \dots, x^{(m)}).$$

Typical choices include coordinatewise $\min$ / $\max$ and rational convex blends $A_{m,\ell}(\vec{x}) = \sum_{r=1}^m \lambda_r^{(\ell)} x^{(r)}$ with $\lambda^{(\ell)} \in \Delta_m \cap \mathbb{Q}^m$. Set

$$\mathcal{MS} = (H, \{\#^{(m)}\}_{m \geq 1}).$$

Reachable multischedules and selection. Given a finite seed set $S_0 \subset H$, let $\text{Cl}_{\mathcal{MS}}(S_0)$ be the smallest subset of H containing S_0 that is closed under taking elements of $\#^{(m)}$ -images and under repeated applications of all $\#^{(m)}$. A *multischedule* is any $M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))$. A selector $\text{Sel} : \mathcal{M}(H) \rightarrow H$ picks a representative schedule (e.g. one minimizing an evaluation below).

Plithogenic optimization. Let $\mathbb{E}_p$ be a plithogenic expectation and Cr_p a plithogenic credibility that reduce to classical counterparts under the embeddings stated later. We consider the canonical PPMS formulations:

$$\begin{aligned} \text{(PPMS-EVM)} \quad & \min_{M \in \mathcal{M}(\text{Cl}_{\mathcal{MS}}(S_0))} \mathbb{E}_p[\tilde{C}(\text{Sel}(M), \tilde{n})] \\ & \text{s.t. } \mathbb{E}_p[\tilde{T}(\text{Sel}(M), \tilde{n})] \leq T_0; \\ \text{(PPMS-Cred)} \quad & \min_{M, C_0} C_0 \\ & \text{s.t. } \text{Cr}_p\{\tilde{T}(\text{Sel}(M), \tilde{n}) \leq T_0\} \geq \beta, \\ & \text{Cr}_p\{\tilde{C}(\text{Sel}(M), \tilde{n}) \leq C_0\} \geq \alpha. \end{aligned}$$

Example 3.19 (Plithogenic Project MultiScheduling — Offshore Turbine Maintenance). An offshore turbine maintenance campaign has attribute-dependent durations due to *sea state*. Let the attribute be $v = \text{sea}$ with values $Pv = \{\text{calm}, \text{rough}\}$ and contradiction $pCF(\text{calm}, \text{rough}) = 0.7, pCF(a, a) = 0$. For the hoist operation $(2, 3)$ (hours),

$$pdf_{23}(x; \text{calm}) = \max\{0, 1 - |x - 8|/2\}, \quad pdf_{23}(x; \text{rough}) = \max\{0, 1 - |x - 12|/3\}.$$

The plithogenic aggregator for combining steps with attribute values a, b is

$$A_c(x, y) = (1 - c) \min\{x, y\} + c \max\{x, y\}, \quad c = pCF(a, b).$$

Two feasible (single) schedules are $x^{(T)} = (0, 0, 0)$ (tight window) and $x^{(R)} = (0, 1, 2)$ (robust slack). The arity-2 multi-operation produces $M = \{x^{(T)}, x^{(R)}, \frac{1}{2}(x^{(T)} + x^{(R)})\}$. Start/finish and completion times propagate via the plithogenic lifts of $+$ and $\max$ using A_c . The planner requires

$$\text{Cr}_p\{\tilde{T}(x, \tilde{n}) \leq 18\} \geq 0.80$$

and minimizes a plithogenic expected cost $\mathbb{E}_p[\tilde{C}(x, \tilde{n})]$ that includes vessel standby premiums under rough seas. Because large pCF drives aggregation toward the t -conorm side, rough/calm disagreements inflate effective durations; the selector therefore chooses the robust schedule (or its blend) from the support of M , demonstrating plithogenic multi-scheduling with attribute-driven uncertainty and contradiction.

Example 3.20 (Plithogenic Project MultiScheduling — Smart-City Streetlight Retrofit). A smart-city team upgrades streetlights in one district. Each block requires two activities: $(1, 2)$ “install LED and controller,” $(2, 3)$ “integrate with IoT platform.” However, the duration of integration depends on a plithogenic attribute

$$v = \text{“network condition”}, \quad Pv = \{\text{fiber-ok}, \text{4G-limited}, \text{5G-spotty}\}.$$

For the integration activity $(2, 3)$ (in hours), define the appurtenance functions

$$pdf_{23}(x; \text{fiber-ok}) = \max\{0, 1 - |x - 2|/0.5\},$$

$$pdf_{23}(x; \text{4G-limited}) = \max\{0, 1 - |x - 3.5|/1\},$$

$$pdf_{23}(x; \text{5G-spotty}) = \max\{0, 1 - |x - 4.5|/1.2\}.$$

Contradiction between attribute values is encoded by

$$pCF(\text{fiber-ok}, \text{4G-limited}) = 0.4,$$

$$pCF(\text{fiber-ok}, \text{5G-spotty}) = 0.8,$$

$$pCF(\text{4G-limited}, \text{5G-spotty}) = 0.6,$$

and $pCF(a, a) = 0$. The plithogenic aggregator for combining two steps with attribute values a, b is

$$A_{pCF(a,b)}(x, y) = (1 - pCF(a, b)) \min\{x, y\} + pCF(a, b) \max\{x, y\}.$$

Two crews propose two single schedules for three consecutive blocks. Let $H = \mathbb{R}_{\geq 0}^3$ represent start times of the “integration” activity for blocks 1,2,3.

$$x^{(F)} = (0, 2, 4) \quad (\text{fast crew, assuming fiber-ok everywhere}),$$

$$x^{(R)} = (0, 3, 6) \quad (\text{robust crew, assuming some 4G-limited areas}).$$

Define the arity-2 multi-operation on schedules by

$$\#^{(2)}(x^{(1)}, x^{(2)}) = \left\{ \min_{\text{coord}}(x^{(1)}, x^{(2)}), \max_{\text{coord}}(x^{(1)}, x^{(2)}), \frac{2}{3}x^{(1)} + \frac{1}{3}x^{(2)}, \frac{1}{3}x^{(1)} + \frac{2}{3}x^{(2)} \right\}.$$

From $(x^{(F)}, x^{(R)})$ we obtain the multischedule

$$M = \left\{ (0, 2, 4), (0, 3, 6), (0, \frac{7}{3}, \frac{14}{3}), (0, \frac{8}{3}, \frac{16}{3}) \right\}.$$

Plithogenic node and completion times are propagated by the plithogenic lifts of $+$ and $\max$ using $A_{pCF(a,b)}$. Suppose the city wants the retrofit of all three blocks to be *plithogenically* completed within $T_0 = 10$ hours with credibility

$$\text{Cr}_p\{\tilde{T}(x, \tilde{n}) \leq 10\} \geq 0.85,$$

and they minimize a plithogenic expected cost $\mathbb{E}_p[\tilde{C}(x, \tilde{n})]$ that includes: (i) night-work premium (higher when the schedule is earlier), (ii) rework premium (higher when pCF is large because mixing fiber-ok with 5G-spotty inflates duration).

Define the selector

$$\text{Sel}(M) \in \arg \min_{x \in \text{supp}(M)} \left\{ \mathbb{E}_p[\tilde{C}(x, \tilde{n})] : \text{Cr}_p\{\tilde{T}(x, \tilde{n}) \leq 10\} \geq 0.85 \right\}.$$

If the actual district has a mix of fiber-ok and 5G-spotty, the high contradiction pCF (fiber-ok, 5G-spotty) = 0.8 pushes the aggregated durations upward for the very fast schedule $(0, 2, 4)$, potentially violating the deadline. One of the blended schedules, for example $(0, \frac{7}{3}, \frac{14}{3})$, better balances attribute contradiction and allows the city to keep the credibility constraint while avoiding high night-work premiums. Thus, the plithogenic project multischeduling model selects a representative schedule that is not only time-feasible but also attribute-consistent under explicit contradiction management.

Proposition 3.21 (The schedule space is a MultiStructure). *For every $m \geq 1$ and $\vec{x} \in H^m$, the image $\#^{(m)}(\vec{x})$ is a finite multiset in $\mathcal{M}(H)$. Hence $\mathcal{MS} = (H, \{\#^{(m)}\}_{m \geq 1})$ is a MultiStructure.*

Proof. Each $\mathcal{A}_m$ is finite; therefore $\{A_{m,\ell}(\vec{x})\}_{\ell=1}^{q_m}$ is a finite subset of H . Counting duplicates yields an element of $\mathcal{M}(H)$. $\square$

Theorem 3.22 (PPMS generalizes Plithogenic Project Scheduling). *Let $S_0 = H$, take the degenerate MultiStructure with $\mathcal{A}_1 = \{\text{Id}\}$ and $\mathcal{A}_m = \emptyset$ for $m \geq 2$, so that $\#^{(1)}(x) = \{x\}$, and set $\text{Sel}(\{x\}) = x$. Then PPMS–EVM (resp. PPMS–Cred) reduces exactly to single–schedule plithogenic project scheduling (EVM, resp. credibility form).*

Proof. With only the identity multi–operation, $\text{Cl}_{\mathcal{MS}}(S_0) = H$ and every feasible multischedule can be taken as a singleton $\{x\}$. Because $\text{Sel}(\{x\}) = x$, both PPMS formulations reduce to optimizing a single $x \in H$ under the plithogenic propagation $\mathbb{E}_p$ and the chosen $\mathbb{E}_p/\text{Cr}_p$, i.e. to the standard plithogenic scheduling model. $\square$

Theorem 3.23 (PPMS generalizes Fuzzy Project MultiScheduling). *Embed each fuzzy duration $\tilde{n}_{ij}^F$ (membership $\mu_{ij} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$) as a plithogenic duration with*

$$s = 1, \quad Pv = \{*\}, \quad pCF \equiv 0, \quad pdf_{ij}(\cdot; *) = \mu_{ij}.$$

Fix $T = \min$, $S = \max$, hence $A_{c=0}(x, y) = \min\{x, y\}$. Assume the reductions $\mathbb{E}_p = \mathbb{E}$ and $\text{Cr}_p = \text{Cr}$ on this embedding. With the same MultiStructure $\mathcal{MS}$ and selector Sel as in FPMS, PPMS–EVM/PPMS–Cred coincide with FPMS (EVM/credibility).

Proof. Since Pv is a singleton and $pCF \equiv 0$, the plithogenic aggregator becomes $A_0 = \min$. Therefore the E_p -lift of $+$ and $\max$ equals Zadeh's extension with t -norm $\min$ and t -conorm $\max$, i.e. the standard fuzzy propagation of durations along paths. By reduction, $\mathbb{E}_p$ and Cr_p return the fuzzy expectation and credibility. Because $\mathcal{MS}$ and Sel are unchanged, feasible multischedules and the selected representative schedules are identical to FPMS, and objective/constraints take the same values. $\square$

Theorem 3.24 (PPMS generalizes Neutrosophic Project MultiScheduling). *Embed each single-valued neutrosophic duration $\tilde{n}_{ij}^N = (T_{ij}, I_{ij}, F_{ij})$ as plithogenic with*

$$s = 3, \quad Pv = \{*\}, \quad pCF \equiv 0, \quad pdf_{ij}(\cdot; *) = (T_{ij}, I_{ij}, F_{ij}).$$

Let E_p act componentwise with $A_0 = \min$ (for truth/indeterminacy) and dual choice for falsity, matching the neutrosophic lift used in NPMS. Assume $\mathbb{E}_p$ and Cr_p reduce to the neutrosophic expectation $\mathbb{E}_N$ and credibility Cr_N on this embedding. With the same MultiStructure $\mathcal{MS}$ and selector Sel as in NPMS, PPMS-EVM/PPMS-Cred coincide with NPMS.

Proof. With Pv singleton and $pCF \equiv 0$, the plithogenic aggregator reduces componentwise to the neutrosophic propagation rules ($\sup$ - $\min$ for T/I and the standard dual for F). Hence node/finish times and costs produced by E_p coincide with those of NPMS. By reduction, $\mathbb{E}_p = \mathbb{E}_N$ and $Cr_p = Cr_N$ on the embedding. Keeping $\mathcal{MS}$ and Sel fixed implies the feasible multischedules and the chosen representative schedules match NPMS exactly, yielding identical optimization problems. $\square$

3.6 Fuzzy Project Iterative MultiScheduling

We formalize an iterative, hierarchical version of fuzzy project multischeduling in which schedules are combined across several levels by MultiStructure operators; feasibility and optimality are judged after flattening the hierarchy and selecting a representative schedule for fuzzy evaluation.

Notation 3.25 (Project and fuzzy times). *Let $G = (V, A)$ be a finite DAG with source 1 and sink $n+1$. A (single) schedule is a vector $x = (x_1, \dots, x_n) \in H := \mathbb{R}_{\geq 0}^n$. Each activity $(i, j) \in A$ has a fuzzy duration $\tilde{n}_{ij}$ (e.g. a normal, convex fuzzy number on $\mathbb{R}_{\geq 0}$). For $x \in H$, define fuzzy node times $\tilde{T}_i(x, \tilde{n})$ and the fuzzy project completion time*

$$\tilde{T}(x, \tilde{n}) = \max_{(k, n+1) \in A} (\tilde{T}_k(x, \tilde{n}) + \tilde{n}_{k, n+1}),$$

by Zadeh's extension principle applied to $(u, v) \mapsto u+v$ and $(u, v) \mapsto \max\{u, v\}$. Let $\tilde{C}(x, \tilde{n})$ be a fuzzy total cost built from x and $\tilde{T}(\cdot)$ (e.g. time-cost accrual). Fix a fuzzy expectation $\mathbb{E}[\cdot]$ and a fuzzy credibility $Cr\{\cdot\}$.

Notation 3.26 (Multiset towers). *Let $\mathcal{M}(X)$ denote the set of finite multisets on X (Definition 2.7). Define the multiset towers*

$$\mathcal{M}^0(H) = H, \quad \mathcal{M}^{i+1}(H) = \mathcal{M}(\mathcal{M}^i(H)) \quad (i \geq 0).$$

Definition 3.27 (Iterative MultiStructure on schedules of order k). *Fix a depth $k \in \mathbb{N}$. An Iterative MultiStructure of order k on H is a family of multi-operations*

$$\mathcal{IMS}^{(k)} = \left(H, \left\{ \#^{(m, i)} : (\mathcal{M}^i(H))^m \rightarrow \mathcal{M}^{i+1}(H) \right\}_{m \in \mathcal{I}, 0 \leq i < k} \right),$$

where $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ is a set of arities. For each level i and arity m , fix a *finite* family of combinators $\mathcal{A}_m^{(i)} = \{A_{m,1}^{(i)}, \dots, A_{m,q_{m,i}}^{(i)}\}$ with $A_{m,\ell}^{(i)} : (\mathcal{M}^i(H))^m \rightarrow \mathcal{M}^i(H)$ (e.g. coordinatewise $\min / \max$ and rational convex combinations when $i = 0$, or their natural multiset lifts for $i \geq 1$), and set

$$\#^{(m, i)}(X^{(1)}, \dots, X^{(m)}) = \{A_{m,1}^{(i)}(\vec{X}), \dots, A_{m,q_{m,i}}^{(i)}(\vec{X})\} \in \mathcal{M}^{i+1}(H), \quad \vec{X} = (X^{(1)}, \dots, X^{(m)}).$$

Definition 3.28 (Flattening (unnesting) operators). For $M \in \mathcal{M}(\mathcal{M}(X))$, define the *flattening* multi-set $\text{Unnest}(M) \in \mathcal{M}(X)$ by multiplicity aggregation:

$$\forall x \in X : \quad m_{\text{Unnest}(M)}(x) = \sum_{A \in \mathcal{M}(X)} m_M(A) \cdot m_A(x).$$

Recursively define $\text{Unnest}^0 = \text{id}_{\mathcal{M}(H)}$ and $\text{Unnest}^{i+1} = \text{Unnest} \circ \text{Unnest}^i$ as maps $\mathcal{M}^{i+2}(H) \rightarrow \mathcal{M}(H)$.

Definition 3.29 (Reachable iterative multischedules and selector). Fix a finite seed set $S_0 \subset H$. Let $\text{Cl}_{\mathcal{IMS}}^{(0)}(S_0)$ be the smallest subset of $\mathcal{M}^0(H) = H$ containing S_0 that is closed under taking members of any $\#^{(m,0)}$ -image and under applications of all $\#^{(m,0)}$. Inductively for $i = 0, 1, \dots, k-1$, define $\text{Cl}_{\mathcal{IMS}}^{(i+1)}(S_0) \subseteq \mathcal{M}^{i+1}(H)$ as the smallest set containing $\#^{(m,i)}((Z^{(1)}, \dots, Z^{(m)}))$ for all $Z^{(r)} \in \text{Cl}_{\mathcal{IMS}}^{(i)}(S_0)$ and all $m \in \mathcal{I}$, and closed under taking elements of multisets. An *iterative multischedule of depth k* is any $M^{(k)} \in \mathcal{M}^k(H)$ that belongs to $\text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)$. Let $\text{Sel} : \mathcal{M}(H) \rightarrow H$ be a deterministic level-1 selector (e.g. the lowest expected fuzzy cost over the support, with fixed tie-breaking). Define the derived selector for depth k by

$$\text{Sel}^{\downarrow k}(M^{(k)}) = \text{Sel}(\text{Unnest}^{k-1}(M^{(k)})) \in H,$$

i.e. flatten down to level 1 and then pick a representative schedule.

Definition 3.30 (Fuzzy Project Iterative MultiScheduling (FPiMS)). Fix $k \in \mathbb{N}$, an order- k Iterative MultiStructure $\mathcal{IMS}^{(k)}$ on H , a seed set $S_0 \subset H$, and a level-1 selector $\text{Sel} : \mathcal{M}(H) \rightarrow H$. The *FPiMS problem of order k* is the optimization over depth- k iterative multischedules:

$$\begin{aligned} \text{(FPiMS-EVM)} \quad & \min_{M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)} \mathbb{E}[\tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n})] \\ \text{s.t.} \quad & \mathbb{E}[\tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n})] \leq T_0, \end{aligned}$$

and, in credibility form,

$$\begin{aligned} \text{(FPiMS-Cred)} \quad & \min_{M^{(k)}, C_0} C_0 \\ \text{s.t.} \quad & \text{Cr}\{\tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq T_0\} \geq \beta, \\ & \text{Cr}\{\tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq C_0\} \geq \alpha. \end{aligned}$$

Example 3.31 (Fuzzy Project Iterative MultiScheduling — Global Product Launch). A consumer-electronics firm plans a three-task pipeline on a chain DAG $G = (V, A)$ with $A = \{(1, 2), (2, 3), (3, 4)\}$: design (1, 2), assembly (2, 3), QA (3, 4). Task durations (days) are triangular fuzzy numbers

$$\tilde{n}_{12} = \text{TFN}(5, 6, 8), \quad \tilde{n}_{23} = \text{TFN}(3, 4, 6), \quad \tilde{n}_{34} = \text{TFN}(2, 3, 4),$$

so the serial completion time at earliest starts is $\tilde{T} = \text{TFN}(10, 13, 18)$ with centroid 13.67.

Level 0 (teams). Two regions propose candidate (single) schedules $x \in H = \mathbb{R}_{\geq 0}^3$. For EMEA: $x_A^E = (0, 0, 0)$ and $x_B^E = (0, 1, 1)$; for AMER: $x_A^A = (0, 0, 0)$ and $x_B^A = (0, 1, 2)$.

Level 1 (department). A multi-operation $\#^{(2,0)}$ combines any pair by outputting the finite multiset $\{x^{(\ell)}, x^{(\ell')}, \frac{1}{2}(x^{(\ell)} + x^{(\ell')})\}$. Thus each region forms a multischedule, e.g. $M_{\text{EMEA}}^{(1)} = \{x_A^E, x_B^E, (0, 0.5, 0.5)\}$ and $M_{\text{AMER}}^{(1)} = \{x_A^A, x_B^A, (0, 0.5, 1)\}$.

Level 2 (corporate). A level-1 combiner $\#^{(2,1)}$ maps any pair of level-1 multischedules to a level-2 multiset of coordinatewise-max merges of their members, producing $M^{(2)}$.

Flatten via Unnest^1 to level-1 and select a representative schedule $x^* = \text{Sel}(\text{Unnest}^1(M^{(2)}))$ that minimizes $\mathbb{E}[\tilde{C}(x, \tilde{n})]$ subject to $\mathbb{E}[\tilde{T}(x, \tilde{n})] \leq T_0$. Under a deadline $T_0 = 15$, Sel typically prefers a blended plan (e.g. $(0, 0.5, 1)$), which lowers staffing penalty while keeping fuzzy completion feasible.

Example 3.32 (Fuzzy Project Iterative MultiScheduling for Seasonal Retail Rollout). Consider a retail company that must roll out a new point-of-sale (POS) subsystem to stores across three regions (North, Central, South) over the next two quarters. The project manager plans several *iterations* (I1 = pilot, I2 = partial rollout, I3 = full rollout) and several *parallel workstreams*: (1) hardware delivery, (2) POS software configuration, (3) staff training, and (4) payment-gateway integration.

However, each task duration is imprecise because it depends on fuzzy factors such as “supplier responsiveness,” “availability of trainers,” and “store traffic”. For example:

$$\mu_{\text{fast supplier}}(\text{North}) = 0.8, \quad \mu_{\text{fast supplier}}(\text{Central}) = 0.5, \quad \mu_{\text{fast supplier}}(\text{South}) = 0.3.$$

Similarly, the training effort has fuzzy availability:

$$\mu_{\text{trainer available}}(\text{week 3}) = 0.6, \quad \mu_{\text{trainer available}}(\text{week 4}) = 0.9.$$

The project schedule for each iteration is therefore not a single crisp Gantt chart but a *fuzzy family of feasible schedules*:

$$\mathcal{S}_{lk} = \{ s \text{ (schedule)} \mid \mu_{\text{feasible}}(s) > 0 \},$$

where μ_{feasible} aggregates fuzzy task durations, fuzzy resource availability, and fuzzy regional readiness by a chosen fuzzy operator (e.g. min, product, or a bounded t-norm).

After iteration I1 (pilot in North) is completed, real execution data are collected (actual delivery time, actual training time, actual weekend traffic). The manager then *updates* the fuzzy membership functions (e.g. narrowing the support of the delivery-time fuzzy number for North) and *re-runs the multischeduling* for iteration I2. Thus the project becomes an *iterative fuzzy multischeduling* process:

$$\text{Plan} \rightarrow \text{Execute} \rightarrow \text{Update Fuzzy Params} \rightarrow \text{Reschedule},$$

where at each step multiple regions and multiple workstreams are co-scheduled under fuzzy uncertainty about time and resource usage.

Proposition 3.33 (Schedule carrier is an Iterative MultiStructure). *For every depth $k \geq 1$, the tuple*

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(m,i)} : (\mathcal{M}^i(H))^m \rightarrow \mathcal{M}^{i+1}(H) \}_{m \in \mathcal{I}, 0 \leq i < k} \right)$$

is an Iterative MultiStructure of order k in the sense of Definition 2.11.

Proof. Each index set $\mathcal{A}_m^{(i)}$ is finite; hence, for any input $(X^{(1)}, \dots, X^{(m)}) \in (\mathcal{M}^i(H))^m$, the image $\#^{(m,i)}(X^{(1)}, \dots, X^{(m)})$ is a finite multiset of elements of $\mathcal{M}^i(H)$ and thus lies in $\mathcal{M}^{i+1}(H)$. This is exactly the typing requirement in Definition 2.11. $\square$

Theorem 3.34 (FPiMS generalizes Fuzzy Project MultiScheduling). *Let $k = 1$. Choose the degenerate level-0 MultiStructure with $\mathcal{A}_1^{(0)} = \{\text{Id}\}$ and $\mathcal{A}_m^{(0)} = \emptyset$ for $m \geq 2$, so that $\#^{(1,0)}(x) = \{x\}$ for all $x \in H$. Set $S_0 = H$ and $\text{Sel}(\{x\}) = x$. Then FPiMS–EVM (resp. FPiMS–Cred) reduces exactly to the Fuzzy Project MultiScheduling model (EVM, resp. credibility form).*

Proof. With $k = 1$ we optimize over depth-1 objects $M^{(1)} \in \text{Cl}_{\mathcal{IMS}}^{(1)}(S_0) \subseteq \mathcal{M}(H)$. Because the only active multi-operation is $\#^{(1,0)}(x) = \{x\}$ and $S_0 = H$, the closure adds nothing: $\text{Cl}_{\mathcal{IMS}}^{(1)}(S_0) = \mathcal{M}(H)$ with all singletons $\{x\}$ ($x \in H$) admissible. Flattening is vacuous: $\text{Unnest}^0 = \text{id}_{\mathcal{M}(H)}$, hence $\text{Sel}^{\downarrow 1}(M^{(1)}) = \text{Sel}(M^{(1)})$ and, in particular, $\text{Sel}(\{x\}) = x$. Consequently, FPiMS–EVM reads

$$\min_{M^{(1)} \in \mathcal{M}(H)} \mathbb{E} \left[\tilde{C}(\text{Sel}(M^{(1)}), \tilde{n}) \right] \quad \text{s.t.} \quad \mathbb{E} \left[\tilde{T}(\text{Sel}(M^{(1)}), \tilde{n}) \right] \leq T_0.$$

Since any optimal multiset can be chosen as a singleton $\{x\}$ (the evaluation depends only on the selected representative), this is equivalent to

$$\min_{x \in H} \mathbb{E}[\tilde{C}(x, \tilde{n})] \quad \text{s.t.} \quad \mathbb{E}[\tilde{T}(x, \tilde{n})] \leq T_0,$$

which is exactly the FPMS (EVM) formulation. The same argument with events $\{\tilde{T}(\cdot) \leq T_0\}$ and $\{\tilde{C}(\cdot) \leq C_0\}$ shows that FPiMS–Cred reduces to FPMS (credibility form). $\square$

3.7 Neutrosophic Project iterative MultiScheduling

Neutrosophic Project Iterative MultiScheduling constructs hierarchical multisets with truth–indeterminacy–falsity durations, iteratively combines and flattens candidates, optimizing neutrosophic expectation and credibility.

Notation 3.35 (Network, schedules, and neutrosophic times). Let $G = (V, A)$ be a finite DAG with source 1 and sink $n+1$, and let $H := \mathbb{R}_{\geq 0}^n$ be the set of (single) schedules $x = (x_1, \dots, x_n)$. For each $(i, j) \in A$, the activity duration is a single-valued neutrosophic number $\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij})$ on $\mathbb{R}_{\geq 0}$. Write E for the componentwise extension principle on the scalar maps $(u, v) \mapsto u+v$ and $(u, v) \mapsto \max\{u, v\}$ introduced earlier. For $x \in H$, the neutrosophic node times $\tilde{T}_i(x, \tilde{n})$ and the project completion time $\tilde{T}(x, \tilde{n})$ are obtained by applying E to $+$ and $\max$ along the precedence constraints. Let $\tilde{C}(x, \tilde{n})$ be a (single-valued) neutrosophic project cost derived from $\tilde{T}(\cdot)$ via E -lifting of a scalar cost model. Let Cr_N and $\mathbb{E}_N$ denote a neutrosophic credibility and a neutrosophic expectation, respectively.

Notation 3.36 ((Recall) Multiset towers and flattening). Let $\mathcal{M}(X)$ be the set of finite multisets on a set X (Definition 2.7). Define the multiset towers $\mathcal{M}^0(H) = H$ and $\mathcal{M}^{i+1}(H) = \mathcal{M}(\mathcal{M}^i(H))$ for $i \geq 0$. For $M \in \mathcal{M}(\mathcal{M}(X))$, let $\text{Unnest}(M) \in \mathcal{M}(X)$ be the flattening that sums multiplicities. Recursively set $\text{Unnest}^0 = \text{id}_{\mathcal{M}(H)}$ and $\text{Unnest}^{i+1} = \text{Unnest} \circ \text{Unnest}^i$.

Definition 3.37 ((Recall) Reachable iterative multischedules and selector). Let $S_0 \subset H$ be finite. Define $\text{Cl}_{\mathcal{IMS}}^{(0)}(S_0)$ as the smallest subset of H that contains S_0 and is closed under taking elements of $\#^{(m,0)}$ -images and under repeated applications of all $\#^{(m,0)}$. For $i = 0, 1, \dots, k-1$, define $\text{Cl}_{\mathcal{IMS}}^{(i+1)}(S_0) \subseteq \mathcal{M}^{i+1}(H)$ as the smallest set containing all $\#^{(m,i)}(Z^{(1)}, \dots, Z^{(m)})$ with $Z^{(r)} \in \text{Cl}_{\mathcal{IMS}}^{(i)}(S_0)$ and closed under taking elements of multisets. An *iterative multischedule of depth k* is any $M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)$. Let $\text{Sel} : \mathcal{M}(H) \rightarrow H$ be a deterministic selector on level 1. Define the derived selector on depth k by

$$\text{Sel}^{\downarrow k}(M^{(k)}) = \text{Sel}(\text{Unnest}^{k-1}(M^{(k)})) \in H.$$

Definition 3.38 (Neutrosophic Project Iterative MultiScheduling (NPiMS)). Fix $k \in \mathbb{N}$, an order- k Iterative MultiStructure $\mathcal{IMS}^{(k)}$ on H , a seed set $S_0 \subset H$, and a selector $\text{Sel} : \mathcal{M}(H) \rightarrow H$. The *NPiMS problem of order k* is the optimization over depth- k iterative multischedules $M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)$:

$$\begin{aligned} \text{(NPiMS–EVM)} \quad & \min_{M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)} \mathbb{E}_N \left[\tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \right] \\ & \text{s.t.} \quad \mathbb{E}_N \left[\tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \right] \leq T_0; \\ \text{(NPiMS–Cred)} \quad & \min_{M^{(k)}, C_0} C_0 \\ & \text{s.t.} \quad \text{Cr}_N \left\{ \tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq T_0 \right\} \geq \beta, \\ & \quad \text{Cr}_N \left\{ \tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq C_0 \right\} \geq \alpha. \end{aligned}$$

Example 3.39 (Neutrosophic Project Iterative MultiScheduling — Emergency Field Hospitals). A city deploys field hospitals through (1, 2) tent erection, (2, 3) power, (3, 4) equipment. Each duration is a single-valued neutrosophic number $\tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij})$ (hours). For power delivery, for instance, T_{23} peaks at 6, I_{23} is 0.3 on $[5, 7]$, and $F_{23} = 1 - T_{23}$ outside that range.

Level 0 (wards). Ward w proposes two schedules $x_A^w, x_B^w \in H = \mathbb{R}_{\geq 0}^3$ (fast vs. buffered). **Level 1 (districts).** A multi-operation $\#^{(m,0)}$ applied to each district's ward set outputs the multiset of all candidates and their rational convex blends. This yields district multischedules $M_d^{(1)}$. **Level 2 (city).** $\#^{(2,1)}$ merges district multischedules by taking feasible coordinatewise-max pairings to avoid crew conflicts, producing a city-level $M^{(2)}$.

Flatten via Unnest^1 and choose $x^* = \text{Sel}(\text{Unnest}^1(M^{(2)}))$ to

$$\min \mathbb{E}_N[\tilde{C}(x, \tilde{n})] \quad \text{s.t.} \quad \text{Cr}_N\left\{\tilde{T}(x, \tilde{n}) \leq 14\right\} \geq 0.85,$$

where times propagate by the componentwise neutrosophic lift of $+$ and $\max$. Buffered options increase credibility while controlling expected overtime, so Sel typically returns a blended buffered plan.

Example 3.40 (Neutrosophic Project Iterative MultiScheduling for University Digitalization). A university is digitalizing student services (admission, course registration, tuition payment, and student support). The project is divided into four iterations: I1 = admission portal, I2 = course registration, I3 = payment, I4 = student support. For each iteration, tasks must be aligned with three stakeholder groups: (IT department, Registrar office, External vendor). Because policies may change suddenly (e.g. new compliance from the Ministry), some activities are not purely uncertain but truly *indeterminate*.

For task “Integrate Admission Portal with Legacy DB” in I1, the project office assigns a neutrosophic duration triplet

$$(T_d, I_d, F_d) = (0.55, 0.30, 0.15),$$

meaning: 55% truth that the task can be finished within 2 weeks, 30% indeterminacy because of unclear legacy documentation, and 15% falsity due to a known risk of vendor delay. Similarly, stakeholder approval for the Registrar office in I2 is modeled as

$$(T_a, I_a, F_a) = (0.40, 0.45, 0.15),$$

indicating a high indeterminate part: requirements are not yet fixed.

The *neutrosophic multischeduling* at iteration k seeks a schedule S_k that maximizes the overall neutrosophic feasibility

$$\text{NF}(S_k) = (T(S_k), I(S_k), F(S_k)),$$

subject to resource and precedence constraints over multiple concurrent subprojects (admission, course, payment). After each iteration is executed, the university observes which parts were really blocked (by policy) and which parts were simply delayed by workload. This observation is then used to *update the indeterminacy component* $I(\cdot)$ for future iterations, for example:

$$I_{k+1}(\text{Registrar approval}) := 0.25 \quad \text{if clarifications were received in iteration } k.$$

In this way, the project evolves by repeatedly (1) scheduling several subprojects in parallel, (2) measuring truth/indeterminacy/falsity of completion, and (3) re-scheduling with reduced indeterminacy.

Proposition 3.41 (The schedule carrier of NPiMS is an Iterative MultiStructure). *For every $k \geq 1$, the tuple*

$$\mathcal{IMS}^{(k)} = \left(H, \left\{ \#^{(m,i)} : (\mathcal{M}^i(H))^m \rightarrow \mathcal{M}^{i+1}(H) \right\}_{m \in \mathcal{I}, 0 \leq i < k} \right)$$

is an Iterative MultiStructure of order k in the sense of Definition 2.11.

Proof. Each level- i family of combiners is finite; thus, for any $(X^{(1)}, \dots, X^{(m)}) \in (\mathcal{M}^i(H))^m$, the image $\#^{(m,i)}(X^{(1)}, \dots, X^{(m)})$ is a finite multiset of elements of $\mathcal{M}^i(H)$, i.e. an element of $\mathcal{M}^{i+1}(H)$. This is precisely the typing requirement in Definition 2.11. $\square$

Theorem 3.42 (NPiMS generalizes Neutrosophic Project MultiScheduling). *Let $k = 1$. Take the degenerate level-0 MultiStructure with $\mathcal{A}_1^{(0)} = \{\text{Id}\}$ and $\mathcal{A}_m^{(0)} = \emptyset$ for $m \geq 2$, hence $\#^{(1,0)}(x) = \{x\}$ for all $x \in H$. Let $S_0 = H$ and $\text{Sel}(\{x\}) = x$. Then NPiMS-EVM (resp. NPiMS-Cred) reduces exactly to the Neutrosophic Project MultiScheduling model (EVM, resp. credibility form).*

Proof. With $k = 1$, feasible objects are $M^{(1)} \in \mathcal{M}(H)$ generated by $\#^{(1,0)}(x) = \{x\}$. Thus $\text{Cl}_{\mathcal{IMS}}^{(1)}(S_0) = \mathcal{M}(H)$ and every singleton $\{x\}$ is admissible. Flattening is vacuous: $\text{Unnest}^0 = \text{id}$, so $\text{Sel}^{\downarrow 1}(M^{(1)}) = \text{Sel}(M^{(1)})$ and $\text{Sel}(\{x\}) = x$. Therefore NPiMS-EVM becomes $\min_{x \in H} \mathbb{E}_N[\tilde{C}(x, \tilde{n})]$ subject to $\mathbb{E}_N[\tilde{T}(x, \tilde{n})] \leq T_0$, and NPiMS-Cred becomes $\min_{x, C_0} C_0$ subject to the neutrosophic credibility constraints on $\tilde{T}$ and $\tilde{C}$. These are exactly the NPMS formulations previously introduced. $\square$

Theorem 3.43 (NPiMS generalizes Fuzzy Project Iterative MultiScheduling). *Embed each fuzzy duration $\tilde{n}_{ij}^F$ (membership μ_{ij}) as an SVNN by*

$$\iota_{F \rightarrow N} : \tilde{n}_{ij}^F \mapsto \tilde{n}_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad T_{ij} = \mu_{ij}, \quad I_{ij} \equiv 0, \quad F_{ij} = 1 - \mu_{ij}.$$

Assume that Cr_N and $\mathbb{E}_N$ reduce on such embeddings to the fuzzy credibility Cr and fuzzy expectation $\mathbb{E}$. Let $\mathcal{IMS}^{(k)}$, S_0 , and Sel be the same as in the FPiMS model. Then NPiMS-EVM and NPiMS-Cred coincide with the corresponding FPiMS formulations.

Proof. Under $\iota_{F \rightarrow N}$, the T -components of all constructed neutrosophic quantities (node times, completion time, and cost) are exactly the fuzzy memberships obtained by Zadeh's extension principle, because E acts componentwise and uses the same sup-min mechanism for T as in the fuzzy case; $I \equiv 0$ and $F = 1 - T$ are preserved by the same lift. Hence for any event of the form $\{\tilde{Z} \leq z\}$, its T -marginal equals the fuzzy membership of $\{Z \leq z\}$. By the reduction property, Cr_N and $\mathbb{E}_N$ return Cr and $\mathbb{E}$ on these embeddings. Since $\mathcal{IMS}^{(k)}$, S_0 , and Sel are unchanged, the feasible depth- k multischedules, their flattenings Unnest^{k-1} , and the selected representatives $\text{Sel}^{\downarrow k}$ coincide with those in FPiMS; consequently the objectives and constraints evaluate to the same real values. $\square$

3.8 Plithogenic Project iterative MultiScheduling

Plithogenic Project Iterative MultiScheduling integrates attribute memberships and contradiction, forms multisets, iteratively combines and flattens schedules, optimizing expectation and credibility.

Notation 3.44 (Plithogenic durations and propagation). *Retain the project network $G = (V, A)$ and the schedule space $H = \mathbb{R}_{\geq 0}^n$. For each $(i, j) \in A$, let $\tilde{n}_{ij}$ be a plithogenic duration as in the Plithogenic Project MultiScheduling (PPMS) section, with the plithogenic lift E_p of $+$ and $\max$ induced by the aggregator A_c (based on a fixed t -norm/ t -conorm and a contradiction degree). Write $\tilde{T}_i(x, \tilde{n})$ and $\tilde{T}(x, \tilde{n})$ for the plithogenic node and completion times computed via E_p , and let $\tilde{C}(x, \tilde{n})$ be the plithogenic total cost. Denote by $\mathbb{E}_p$ and Cr_p a plithogenic expectation and credibility, respectively, consistent with the chosen plithogenic model.*

Notation 3.45 (Iterative MultiStructure, flattening, closure, and selection). *Let $\mathcal{IMS}^{(k)}$ be an Iterative MultiStructure of order k on H as in Definition 2.11, acting levelwise on the multiset tower $\{\mathcal{M}^i(H)\}_{i \geq 0}$. Use the flattening operators Unnest^ℓ and the closure families $\text{Cl}_{\mathcal{IMS}}^{(i)}(S_0)$ built from a finite seed set $S_0 \subset H$ as in Definition 3.29. Let $\text{Sel} : \mathcal{M}(H) \rightarrow H$ be a deterministic level-1 selector and let the derived selector be*

$$\text{Sel}^{\downarrow k}(M^{(k)}) = \text{Sel}(\text{Unnest}^{k-1}(M^{(k)})) \in H.$$

Definition 3.46 (Plithogenic Project Iterative MultiScheduling (PPiMS)). Fix $k \in \mathbb{N}$, an order- k Iterative MultiStructure $\mathcal{IMS}^{(k)}$ on H , a finite seed $S_0 \subset H$, and a level-1 selector Sel . The *PPiMS problem of order k* is the following optimization over depth- k iterative multischedules $M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)$:

$$\begin{aligned} \text{(PPiMS-EVM)} \quad & \min_{M^{(k)} \in \text{Cl}_{\mathcal{IMS}}^{(k)}(S_0)} \mathbb{E}_p \left[\tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \right] \\ \text{s.t.} \quad & \mathbb{E}_p \left[\tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \right] \leq T_0, \end{aligned}$$

and, in credibility form,

$$\begin{aligned} \text{(PPiMS-Cred)} \quad & \min_{M^{(k)}, C_0} C_0 \\ \text{s.t.} \quad & \text{Cr}_p \{ \tilde{T}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq T_0 \} \geq \beta, \\ & \text{Cr}_p \{ \tilde{C}(\text{Sel}^{\downarrow k}(M^{(k)}), \tilde{n}) \leq C_0 \} \geq \alpha. \end{aligned}$$

Example 3.47 (Plithogenic Project Iterative MultiScheduling — Offshore Wind Maintenance). A maintenance campaign's durations depend on the attribute $v = \text{sea state}$ with values $Pv = \{\text{calm}, \text{rough}\}$ and contradiction $pCF(\text{calm}, \text{rough}) = 0.6$, $pCF(a, a) = 0$. For the hoist (2, 3) (hours),

$$pdf_{23}(x; \text{calm}) = \max\{0, 1 - |x - 8|/2\}, \quad pdf_{23}(x; \text{rough}) = \max\{0, 1 - |x - 12|/3\}.$$

The plithogenic aggregator for combining steps with attribute values a, b is $A_c(x, y) = (1-c) \min\{x, y\} + c \max\{x, y\}$ with $c = pCF(a, b)$.

Level 0 (turbines). Each turbine t offers two feasible schedules $x_A^t, x_B^t \in H = \mathbb{R}_{\geq 0}^3$. **Level 1 (arrays).** Apply $\#^{(m,0)}$ to turbine sets to form array multischedules $M_{\text{arr}}^{(1)}$ that include candidates and convex blends. Durations propagate via the plithogenic lifts of $+$ and $\max$ using A_c . **Level 2 (farm).** $\#^{(2,1)}$ merges array multischedules into a farm-level $M^{(2)}$ (coordinatewise-max to respect shared vessel availability).

Flatten with Unnest^1 and select $x^* = \text{Sel}(\text{Unnest}^1(M^{(2)}))$ to

$$\min \mathbb{E}_p \left[\tilde{C}(x, \tilde{n}) \right] \quad \text{s.t.} \quad \text{Cr}_p \left\{ \tilde{T}(x, \tilde{n}) \leq 18 \right\} \geq 0.80.$$

Because large contradiction drives aggregation toward the t -conorm side under mixed sea states, robust/blended schedules dominate, and the selector chooses a plan with deliberate slack.

Example 3.48 (Plithogenic Project Iterative MultiScheduling for Smart-Home Product Line). A technology company is developing a smart-home product line consisting of (1) smart hub, (2) energy monitor, (3) security camera, (4) AI-based mobile app. All four components must be released in a coordinated manner, but each task has multiple *attributes* that may *agree* or *contradict*:

- cost priority (low, medium, high),
- security level (basic, advanced),
- time-to-market urgency (urgent, normal),
- integration complexity (simple, complex).

For the task “Implement end-to-end encryption in camera module”, the team assigns a plithogenic membership

$$\mu_{\text{plitho}}(\text{task} \mid \text{security} = \text{advanced}) = 0.9,$$

but with a contradiction degree

$$\text{Contr}(\text{security}=\text{advanced}, \text{cost}=\text{low}) = 0.7,$$

because advanced security typically raises cost. Similarly, for “Develop mobile AI features”, the attributes “time-to-market = urgent” and “integration complexity = complex” have a contradiction degree 0.6.

In plithogenic project iterative multischeduling, the schedule for an iteration (I1 = core hub; I2 = camera + energy; I3 = unified mobile app) is obtained by aggregating tasks’ plithogenic memberships with respect to the dominant attribute of that iteration. For example, in I1 the dominant attribute is “time-to-market = urgent”, so tasks whose attribute set contradicts this (e.g. high complexity, high cost) receive a *lower aggregated membership* in the schedule. Formally, for a task x with attribute value v and dominant attribute value v_d ,

$$\mu^*(x) = \text{PlithoAgg}(\mu(x | v), \text{Contr}(v, v_d)),$$

where the aggregation reduces $\mu(x | v)$ proportionally to the contradiction.

After executing I1, the company observes market feedback: customers accept a slightly higher price but insist on advanced security. This *changes the dominant attribute* for I2 to “security = advanced”, thereby lowering the contradiction for camera-related tasks and raising their plithogenic membership in the next multischeduling run. Thus, at every iteration the project office:

1. re-evaluates attribute dominance from market/manager feedback;
2. re-computes contradiction degrees with respect to the new dominant attribute;
3. re-runs the multischeduling over parallel component teams;
4. locks the next release window.

This captures a realistic situation where several product components are co-developed, but attribute-level contradictions (cost vs. security vs. speed) must be negotiated iteratively in the schedule.

Proposition 3.49 (Schedule carrier of PPIMS is an Iterative MultiStructure). *If, for each level i and arity m , the combiner family defining $\#^{(m,i)}$ is finite, then*

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(m,i)} : (\mathcal{M}^i(H))^m \rightarrow \mathcal{M}^{i+1}(H) \}_{m \in \mathcal{I}, 0 \leq i < k} \right)$$

is an Iterative MultiStructure of order k in the sense of Definition 2.11.

Proof. For any $(X^{(1)}, \dots, X^{(m)}) \in (\mathcal{M}^i(H))^m$, the finite combiner family produces a finite set of outputs in $\mathcal{M}^i(H)$; counting multiplicities yields an element of $\mathcal{M}^{i+1}(H)$. This verifies the typing requirement at every level i and arity m . $\square$

Theorem 3.50 (PPIMS generalizes PPMS). *Let $k = 1$, take the degenerate level-0 structure with $\#^{(1,0)}(x) = \{x\}$ and no higher-arity operators, set $S_0 = H$, and choose $\text{Sel}(\{x\}) = x$. Then (PPIMS–EVM) (resp. (PPIMS–Cred)) reduces exactly to the single-level Plithogenic Project MultiScheduling model (EVM, resp. credibility form).*

Proof. With $k = 1$, the feasible objects are $M^{(1)} \in \mathcal{M}(H)$ and flattening is vacuous: $\text{Sel}^{+1}(M^{(1)}) = \text{Sel}(M^{(1)})$. Because any optimum can be taken at a singleton $\{x\}$ and $\text{Sel}(\{x\}) = x$, both formulations reduce to optimizing a single schedule x under the plithogenic propagation E_p with $\mathbb{E}_p$ and Cr_p , i.e., to PPMS. $\square$

Theorem 3.51 (PPIMS generalizes Fuzzy Project Iterative MultiScheduling). *Embed each fuzzy duration $\tilde{n}_{ij}^F$ (membership μ_{ij}) as a plithogenic duration with*

$$s = 1, \quad Pv = \{*\}, \quad pCF \equiv 0, \quad pdf_{ij}(\cdot; *) = \mu_{ij},$$

and fix t -norm $T = \min$, t -conorm $S = \max$, so $A_{c=0} = \min$. Assume reductions $\mathbb{E}_p = \mathbb{E}$ and $\text{Cr}_p = \text{Cr}$ on this embedding. Use the same Iterative MultiStructure $\mathcal{IMS}^{(k)}$, seed S_0 , and selector Sel as in Fuzzy Project Iterative MultiScheduling (FPiMS). Then (PPiMS–EVM/Cred) coincides with FPiMS (EVM/credibility).

Proof. With P_v a singleton and $pCF \equiv 0$, the plithogenic lift E_p reproduces Zadeh's extension (sup–min for $+$ and $\max$). Hence the propagated node/finish times and costs match the fuzzy ones. By reduction, expectations and credibilities agree. Because $\mathcal{IMS}^{(k)}$, S_0 , and Sel are unchanged, feasible (iterative) multischedules and their selected representatives are identical in both models, yielding identical optimization problems. $\square$

Theorem 3.52 (PPiMS generalizes Neutrosophic Project Iterative MultiScheduling). *Embed each single-valued neutrosophic duration $\tilde{n}_{ij}^N = (T_{ij}, I_{ij}, F_{ij})$ as plithogenic with*

$$s = 3, \quad P_v = \{*\}, \quad pCF \equiv 0, \quad pdf_{ij}(\cdot; *) = (T_{ij}, I_{ij}, F_{ij}),$$

and let E_p act componentwise so that, for T and I , $A_0 = \min$ (with the standard dual for F), thereby reproducing the neutrosophic lift used in Neutrosophic Project iterative MultiScheduling. Assume reductions $\mathbb{E}_p = \mathbb{E}_N$ and $\text{Cr}_p = \text{Cr}_N$ on this embedding. Use the same $\mathcal{IMS}^{(k)}$, S_0 , and Sel as in the neutrosophic model. Then (PPiMS–EVM/Cred) coincides with Neutrosophic Project iterative MultiScheduling (EVM/credibility).

Proof. Under the stated embedding and componentwise aggregation, the plithogenic propagation of $+$ and $\max$ produces the same node/finish times and costs as the neutrosophic extension. Reductions of expectation/credibility transfer objective and constraints verbatim. Keeping $\mathcal{IMS}^{(k)}$, S_0 , and Sel fixed preserves the feasible iterative multischedules and selected representatives, so the optimization problems are identical. $\square$

4. Conclusion

This paper developed a unified mathematical framework for project scheduling under uncertainty based on Iterative MultiStructures. In particular, we introduced Fuzzy, Neutrosophic, and Plithogenic Project Iterative MultiScheduling, in which multiple candidate schedules can be organized, combined, and evaluated across successive hierarchical levels. By representing scheduling alternatives as finite multisets and employing canonical flattening and selection mechanisms, the proposed framework provides a systematic approach to handling both uncertainty and hierarchical multiplicity in complex project environments. The resulting models extend conventional project scheduling and multischeduling by allowing uncertain information and multiple competing schedules to be propagated and reconsidered through iterative planning stages.

Several directions remain open for further investigation. One promising direction is to incorporate Upside-Down Logic [40, 41] to model inversion, reversal, or counterintuitive relationships that may arise during iterative decision processes. Further extensions may also be developed using superhyperstructures [42] and HyperFuzzy Sets [43]. These frameworks may provide richer representations of multiway dependencies, hierarchical interactions, and higher-order uncertainty. Future studies may additionally investigate computational algorithms, complexity analysis, numerical experiments, and practical applications of the proposed scheduling models.

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Conflicts of Interest

The authors declare no conflicts of interest.

Declaration of Generative AI and AI-Assisted Technologies in the Manuscript Preparation Process

During the preparation of this work, the authors used AI-assisted tools, ChatGPT-4, to enhance readability and improve language structure. These tools were not used to generate scientific content, analyses, or conclusions. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the final manuscript. We did not use generative AI tools to create or modify the study figures or artwork.

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