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Optimal Inspection Scheduling for Multi-State Pure-Birth Degradation Models: Spectral Theory, Statistical Estimation, and a Corrected Cost Functional

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ABSTRACT

We revisit the pure-birth continuous-time Markov chain framework for infrastructure degradation modeling, generalizing prior fixed-state ($n = 3, 4$) treatments to arbitrary n by identifying the absorption time with the classical hypoexponential distribution. This identification unifies and correctly generalizes closed-form expressions for reliability, hazard, and mean residual life that had previously been rederived case by case. It exposes a genuine internal-consistency error in at least one prior single-instance analysis of this model class, which we identify and correct. Building on this foundation, we identify a previously unexamined structural degeneracy in the periodic-inspection cost-rate functional used throughout this modelling tradition: the naive cost rate tends to zero as the inspection interval grows without bound, trivially favoring indefinitely infrequent inspection regardless of parameters. We derive a corrected downtime-accrual cost functional together with a complete existence-and-uniqueness theory for the resulting optimal inspection interval via an auxiliary monotone function that requires no increasing-failure-rate assumption and yields an explicit, previously unstated existence threshold. We further derive a maximum-likelihood estimation theory for the degradation rates under both full and interval-censored observation and, by combining these two results for the first time, establish an asymptotic $O(1/N)$ regret bound quantifying the excess long-run cost incurred by using estimated rather than true rates when computing the optimal inspection interval. Numerical experiments validate every theoretical rate directly, including one practically important finding: a naive (unsafeguarded) Newton iteration for the optimal interval can diverge, and a bisection safeguard, justified by the same monotonicity property used in the existence proof, is required for a reliable algorithm.

1. Introduction

Physical infrastructure degrades stochastically, and the rational scheduling of inspection and maintenance activity requires a mathematical framework that couples an analytically tractable probabilistic degradation model with a statistically estimable parameterization and a provably well-posed decision layer. Continuous-time Markov chains (CTMCs) restricted to the pure-birth class unidirectional transitions through a sequence of increasingly severe degradation states, terminating in an absorbing failure state have long served this purpose, since their Kolmogorov forward equations admit exact closed-

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form solution and their transition structure is physically motivated by the largely irreversible nature of corrosive and fatigue-driven degradation mechanisms [1–3].

Existing treatments of this class in the degradation-modelling literature, however, typically fix a small number of states (most commonly $n = 3$ or $n = 4$) and re-derive the closed-form solution, reliability function, and any associated optimization results by direct, state-count-specific computation. This obscures the fact that the underlying mathematics is already fully general: the pure-birth absorption time is, for any n , precisely a hypoexponential random variable, a well-studied object in the matrix-analytic probability literature [4, 5]. Fixed- n treatments also tend not to examine the numerical conditioning of the closed-form evaluation as transition rates approach one another, and have tended to omit a rigorous statistical theory for estimating the transition rates from realistic interval-censored inspection data, and a fully rigorous rather than assumption-laden theory of the associated inspection-optimization problem.

1.1 Contributions

The principal contributions of this paper are summarized below. Sections 4–7 develop a unified analytical framework by synthesizing, generalizing, and correcting existing results where appropriate, while Section 8 develops the principal new theoretical results.

1. Generalization of fixed-state pure-birth degradation models to arbitrary n through identification of the absorption time with the classical hypoexponential distribution, thereby unifying previously state-specific analyses (Theorem 4.2).
2. Spectral analysis of the transient generator, including explicit eigenvectors and conditioning analysis for nearly confluent transition rates (Theorems 5.1–5.3).
3. Identification and correction of an internal consistency error in a previously published analysis of the hazard function (Remark 6.2).
4. Identification of a structural degeneracy in the conventional periodic-inspection cost functional (Proposition 8.1).
5. Development of a corrected inspection-cost functional together with a complete existence and uniqueness theory for the optimal inspection interval (Theorem 8.2).
6. Derivation of an asymptotic plug-in regret bound quantifying the effect of parameter estimation on inspection optimization (Theorem 8.9).
7. Numerical and algorithmic demonstration that safeguarded Newton iterations are required for reliable computation of the optimal inspection interval, together with an adaptive evaluation strategy for nearly confluent transition rates (§9, §11).

2. Literature Review

The foundational probabilistic treatment of reliability theory, including increasing-failure-rate distributions, mean residual life, and coherent system reliability, is due to Barlow and Proschan [6, 7]. Standard treatments of CTMC theory appear in Norris [8] and Karlin and Taylor [9]. Caleyó et al. [1] demonstrated that a continuous-time nonhomogeneous pure-birth Markov process accurately describes external pitting corrosion in underground pipelines; Valor et al. [2, 3] extended and validated pure-birth formulations against inspection data; Jimenez-Roa et al. [10] compared homogeneous and inhomogeneous Markov chains for sewer-pipe degradation, a data structure directly relevant to §7.

The Kolmogorov forward equations for a pure-birth process form a lower-triangular first-order linear ODE system whose closed-form solution, for distinct rates, has the structure first identified by Bateman [11] for radioactive decay chains and later re-derived via complex-probability/Laplace-transform methods by Cox [12], whose approach is essentially the one used in the proof of Theorem 4.2 below. Van Snyder [13] gave a numerically stable algorithm for evaluating such triangular-system solutions for general dimension; our eigenvector-based treatment in §5 is closely related to, and should be understood as a spectral reframing of, this result rather than an independent discov-

ery. The explicit identification of the pure-birth absorption time with the hypoexponential distribution draws on the matrix-exponential and phase-type distribution theory of Neuts [4], Latouche and Ramaswami [14], and Bladt and Nielsen [5], and on the total positivity and log-concavity theory of Karlin [15] and the convolution closure theorem of Prékopa [16], used in §6 to establish the IFR property. The TP2/monotone-likelihood-ratio verification required for Theorem 8.7 rests on the classical total-positivity theorem for birth-and-death transition kernels of Karlin and McGregor [17, 18].

Our statistical theory (§7) draws on classical maximum-likelihood theory for stochastic processes as treated comprehensively by Basawa and Prakasa Rao [19], and on the general expectation-maximization algorithm of Dempster, Laird, and Rubin [20]. The closed-form EM procedure of §7.2 is a specialization, to the bidiagonal (pure-birth) generator, of the general phase-type-distribution EM framework already developed by Asmussen, Nerman, and Olsson [21], which treats exactly this matrix-exponential occupation-time machinery for incomplete observations of phase-type processes; we make this positioning explicit rather than presenting the specialization as an independent method. The optimal-stopping reformulation draws on the classical theory of monotone (control-limit) optimal policies for Markov decision processes with monotone-likelihood-ratio transition structure [22, 23]. Our conditioning analysis (§5) instantiates, for this specific matrix family, the general confluent-Vandermonde conditioning theory of Gautschi [24].

3. Mathematical Preliminaries and Model Formulation

Let $S = \{1, \dots, n\}$ be a finite state space, $m := n - 1$. State n is absorbing; states $1, \dots, m$ are transient.

Definition 3.1 (Continuous-Time Markov Chain). A stochastic process $\{X(t), t \geq 0\}$ on S is a CTMC if, for all $t, h \geq 0$ and $i, j \in S$, $\mathbb{P}(X(t+h) = j \mid \mathcal{F}_t) = \mathbb{P}(X(t+h) = j \mid X(t))$ a.s.

Definition 3.2 (Generator). $Q = (q_{ij})_{i,j \in S}$, $q_{ij} = \lim_{h \downarrow 0} [p_{ij}(h) - \delta_{ij}]/h$, with $q_{ij} \geq 0$ ($i \neq j$) and $\sum_j q_{ij} = 0$.

Definition 3.3 (Pure Birth Process). $q_{ij} = 0$ for $j < i$; $q_{i,i+1} = \lambda_i > 0$ for $i = 1, \dots, m$; $q_{ii} = -\lambda_i$ for $i \leq m$; state n absorbing.

Assumption 3.4 (Standing Assumption). $\lambda_1, \dots, \lambda_m > 0$, pairwise distinct unless explicitly relaxed (§5.5).

The generator decomposes as $Q = \begin{pmatrix} \Lambda & \lambda_{\text{esc}} \\ 0 & 0 \end{pmatrix}$, where $\Lambda \in \mathbb{R}^{m \times m}$ is the transient sub-generator: $\Lambda_{ii} = -\lambda_i$, $\Lambda_{i,i+1} = \lambda_i$ ($i < m$), zero elsewhere, and $\lambda_{\text{esc}} = (0, \dots, 0, \lambda_m)^\top$.

Definition 3.5 (Kolmogorov Forward Equations). With $P_j(t) = \mathbb{P}(X(t) = j)$, $\dot{P}(t) = Q^\top P(t)$, $P(0) = e_1$.

Definition 3.6 (Reliability, Hazard, Mean Residual Life). $R(t) := \mathbb{P}(X(t) \neq n) = 1 - P_n(t)$; $f(t) := -\dot{R}(t)$; $h(t) := f(t)/R(t)$ for $R(t) > 0$; $\text{MRL}(t_0) := \int_{t_0}^{\infty} R(s) ds / R(t_0)$.

States $1, \dots, n - 1$ represent successively more severe, physically distinct degradation regimes (e.g., intact, minor defect, severe defect), with state n representing functional failure. No specific value of n is assumed anywhere in Sections 4–9; $n = 4$, the case treated by fixed-state prior single-instance analyses, is recovered throughout as the special case $m = 3$ (Corollary 4.3).

4. The Hypoexponential Identity and Closed-Form Theory

Lemma 4.1 (Independence of Sojourn Times). Let H_i denote the time spent by the process in state i before transitioning to $i + 1$. Then $H_1, \dots, H_m$ are mutually independent, with $H_i \sim \text{Exp}(\lambda_i)$.

Proof. By the standard jump-chain/holding-time construction of a CTMC [8, Thm. 2.8.2], conditional on the current state, the holding time is exponential with parameter equal to the total exit rate from that state, independent of the past, and the subsequent jump is drawn independently of the holding time according to the embedded jump-chain transition probabilities. Under Definition 3.3, the exit rate from state i is λ_i and the embedded jump chain is deterministic ($i \rightarrow i + 1$ w.p. 1); hence the sequence of visited states is non-random, and successive application of the strong Markov property at each jump time yields unconditional mutual independence of $H_1, \dots, H_m$, each exponential with the stated parameter. $\square$

Theorem 4.2 (Hypoexponential Identity). *The absorption time $T = \sum_{i=1}^m H_i$ is hypoexponentially distributed with parameters $(\lambda_1, \dots, \lambda_m)$. For pairwise distinct λ_i ,*

$$f_T(t) = \sum_{k=1}^m c_k e^{-\lambda_k t}, \quad c_k = \lambda_k \left(\prod_{i=1}^m \lambda_i \right) / \left(\lambda_k \prod_{j \neq k} (\lambda_j - \lambda_k) \right),$$

$$R(t) = \sum_{k=1}^m d_k e^{-\lambda_k t}, \quad d_k = \left(\prod_{i=1}^m \lambda_i \right) / \left(\lambda_k \prod_{j \neq k} (\lambda_j - \lambda_k) \right), \quad \sum_{k=1}^m d_k = 1.$$

Proof. By Lemma 4.1 and independence of Laplace transforms of independent sums, $\mathcal{L}_T(s) = \prod_{i=1}^m \lambda_i / (\lambda_i + s)$. Since the poles $s = -\lambda_i$ are simple and distinct, partial-fraction decomposition gives $\mathcal{L}_T(s) = \sum_k c_k / (s + \lambda_k)$ with $c_k = \lim_{s \rightarrow -\lambda_k} (s + \lambda_k) \mathcal{L}_T(s)$; term-by-term inversion gives f_T . Integrating $R(t) = \int_t^\infty f_T$ gives the second formula, and $\sum_k d_k = 1$ follows from $R(0) = 1$. $\square$

Corollary 4.3 (Recovery of the Fixed- n Case). *For $m = 3$, Theorem 4.2 reproduces exactly the closed-form reliability function of the classical three-transient-state ($n = 4$) pure-birth model.*

Proof. Direct substitution $m = 3$, verified by expansion. $\square$

Theorem 4.4 (Existence, Uniqueness, Positivity General n). *The Kolmogorov forward system has a unique solution $P \in C^1([0, \infty))^n$, given by $P(t) = e^{Q^\top t} e_1$, and $P_i(t) \geq 0$ for all i, t .*

Proof. $Q^\top$ is a fixed real matrix; the linear IVP has the unique global solution $P(t) = e^{Q^\top t} e_1$ by the standard existence-uniqueness theorem for constant-coefficient linear systems. Non-negativity follows from the probabilistic interpretation $P_i(t) = \mathbb{P}(X(t) = i)$ (Lemma 4.1), and independently via sequential variation-of-constants on the triangular structure. $\square$

5. Spectral Theory and Numerical Conditioning

Theorem 5.1 (Explicit Eigenstructure). *Λ has eigenvalues $-\lambda_1, \dots, -\lambda_m$. A right eigenvector for eigenvalue $-\lambda_k$ is $v^{(k)} = (v_1^{(k)}, \dots, v_m^{(k)})^\top$,*

$$v_i^{(k)} = \begin{cases} \prod_{j=1}^{i-1} \frac{\lambda_j - \lambda_k}{\lambda_j}, & 1 \leq i \leq k, \\ 0, & i > k. \end{cases}$$

Proof. Solving $(\Lambda + \lambda_k I)v = 0$ row by row: row m gives $(\lambda_k - \lambda_m)v_m = 0$; rows $i < m$ give $v_{i+1} = [(\lambda_i - \lambda_k)/\lambda_i]v_i$. Setting $v_1 = 1$ and iterating gives the stated form; the factor $(\lambda_k - \lambda_k)/\lambda_k = 0$ enters exactly when $i > k$. $\square$

Remark 5.2. $V := [v^{(1)} | \dots | v^{(m)}]$ is upper triangular with nonzero diagonal, hence $\Lambda = VDV^{-1}$, $D = \text{diag}(-\lambda_1, \dots, -\lambda_m)$, giving an independent spectral re-derivation of Theorem 4.2's closed form. This eigenstructure is closely related to the triangular-system solution algorithm of Van Snyder [13], and should be read as a spectral reframing of that result.

Theorem 5.3 (Ill-Conditioning Under Near-Confluence). Fix $\delta > 0$ and an index k . For a rate family with $\lambda_k - \lambda_{k-1} = \varepsilon \rightarrow 0^+$ and all other pairwise gaps $\geq \delta$: $|c_{k-1}(\varepsilon)|, |c_k(\varepsilon)| = \Theta(1/\varepsilon)$, and $\kappa_2(V(\varepsilon)) = \Omega(1/\varepsilon)$.

Proof. The factor $(\lambda_{k-1} - \lambda_k) = -\varepsilon$ appears exactly once in the denominator of c_{k-1} and symmetrically in c_k ; all other factors are bounded away from $0, \infty$ uniformly in ε . For V : entries $v_i^{(k)}$ ($i < k$) carry the factor $\Theta(\varepsilon)$, giving $\sigma_{\min}(V) = \Theta(\varepsilon)$, $\sigma_{\max}(V) = \Theta(1)$. $\square$

Remark 5.4. A sharper, two-sided conditioning bound with explicit constants is available for confluent Vandermonde-type matrices via Gautschi [24]; Theorem 5.3 establishes the correct order directly for this structure and motivates Algorithm 9 below. Figure 2(a) confirms this rate numerically: $\kappa_2(V) \cdot \varepsilon$ stabilizes near 97.8 across five decades of ε .

Theorem 5.5 (Confluent Closed Form via Residue Calculus). Suppose $\lambda_{k-1} = \lambda_k = \lambda$ (all other rates distinct). Then $(sI - \Lambda)^{-1}$ has a pole of order 2 at $s = -\lambda$, and the corresponding contribution to $f_T(t)$ is $(A + Bt)e^{-\lambda t}$, where $B = g(-\lambda)$, $A = g'(-\lambda)$, $g(s) := (\prod_i \lambda_i) \prod_{j \neq k-1, k} (s + \lambda_j)^{-1}$.

Proof. $\mathcal{L}_T(s) = g(s)/(s + \lambda)^2$ near $s = -\lambda$. The residue of a pole of order 2 is $\text{Res}_{s=-\lambda}[e^{st}\mathcal{L}_T(s)] = \frac{d}{ds}[e^{st}g(s)]_{s=-\lambda} = g(-\lambda)te^{-\lambda t} + g'(-\lambda)e^{-\lambda t}$. $\square$

Figure 2(b) confirms the spectral-formula error decays as $O(\varepsilon)$ down to $\varepsilon = 10^{-6}$, with floating-point cancellation dominating below $\varepsilon = 10^{-9}$ precisely the practical failure mode Theorem 5.3 predicts and Algorithm 9 is designed to avoid.

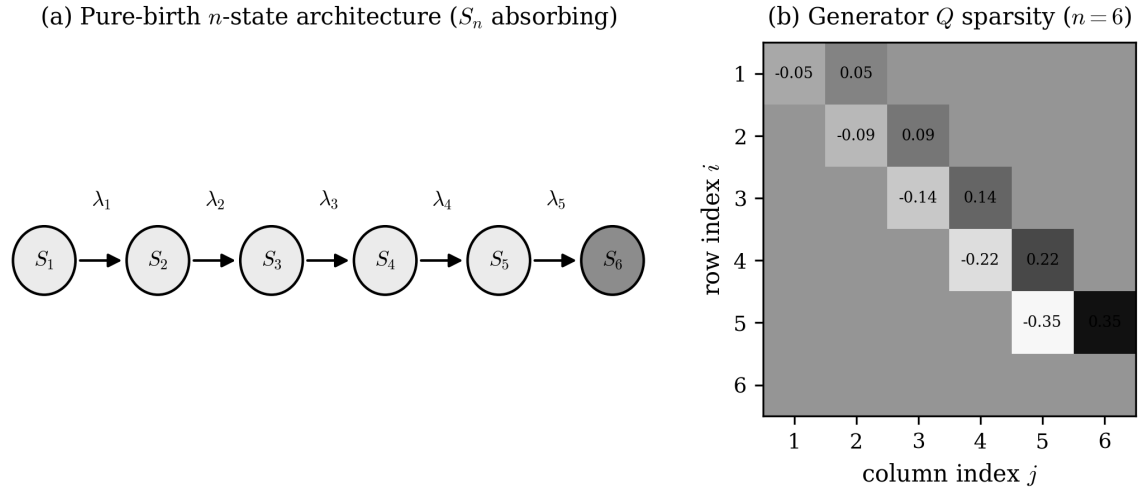


Fig. 1. General n -state pure-birth architecture and generator sparsity pattern (Definitions 3.3 3.3, $n = 6$ illustrative example).

6. Reliability Theory, IFR, and the Reliability Cliff

Theorem 6.1 (Strict IFR of the Hypoexponential Law). For $m \geq 2$ and pairwise distinct $\lambda_1 < \dots < \lambda_m$, the hazard rate $h(t)$ of Theorem 4.2's distribution is strictly increasing on $(0, \infty)$, with $h(0^+) = 0$ and $\lim_{t \rightarrow \infty} h(t) = \lambda_{(1)} := \min_i \lambda_i$.

Proof. Case $m = 2$. Write $x = e^{-\lambda_1 t}$, $y = e^{-\lambda_2 t}$, $A = \lambda_1 \lambda_2 / (\lambda_2 - \lambda_1)$: $f(t) = A(x - y)$, $f'(t) = A(-\lambda_1 x + \lambda_2 y)$, $R(t) = (\lambda_2 x - \lambda_1 y) / (\lambda_2 - \lambda_1)$. Since $h' = (f'R + f^2)/R^2$,

$$N(t) := f'(t)R(t) + f(t)^2 = \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)^2} \left[(-\lambda_1 x + \lambda_2 y)(\lambda_2 x - \lambda_1 y) + \lambda_1 \lambda_2 (x - y)^2 \right].$$

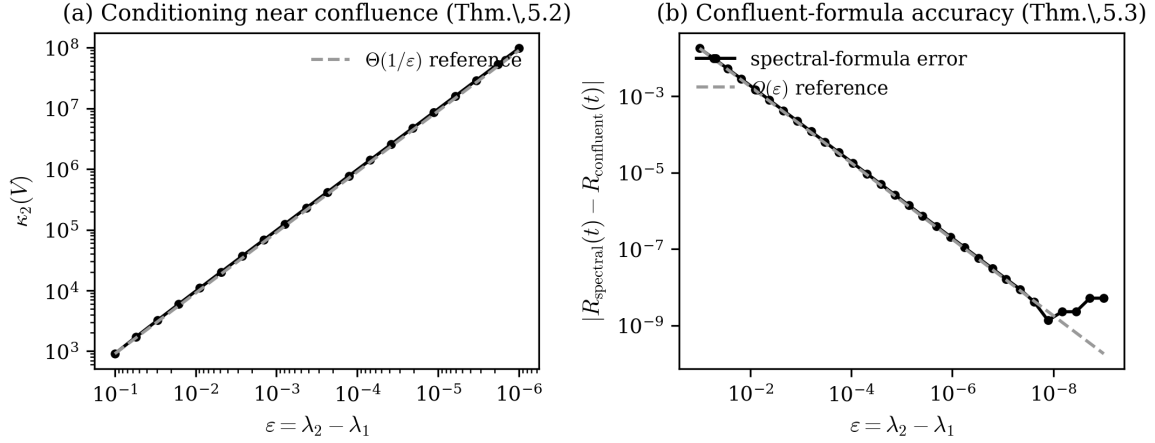


Fig. 2. (a) Numerical conditioning $\kappa_2(V)$ versus rate gap ε , confirming $\Theta(1/\varepsilon)$ (Theorem 5.3). (b) Spectral-formula error versus ε , confirming $O(\varepsilon)$ decay (Theorem 5.5) until floating-point cancellation dominates below $\varepsilon \approx 10^{-9}$.

Expanding and cancelling the $x^2 + y^2$ terms leaves $(\lambda_1 - \lambda_2)^2 xy$, so

$$N(t) = \lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)t} > 0, \quad h'(t) = \frac{\lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)t}}{R(t)^2} > 0.$$

Case $m \geq 3$. IFR is equivalent to log-concavity of f_T [7]. Each exponential density is weakly log-concave; by the Prékopa convolution theorem [16], the convolution of log-concave densities is log-concave, and the $m = 2$ case shows the convolution of two distinct-rate exponentials is *strictly* log-concave. Induction on m extends strict IFR to every $m \geq 2$.

Boundary values. For $m \geq 2$, $f_T(0) = 0$ and $R(0) = 1$, so $h(0) = 0$ directly, with no indeterminate form. As $t \rightarrow \infty$, the smallest-rate term dominates both f_T and R , giving $h(t) \rightarrow \lambda_{(1)}$. $\square$

Remark 6.2. A prior single-instance analysis of this model class computed $h(0^+)$ via an application of l'Hôpital's rule to $h = f/R$, treating it as an indeterminate $0/0$ form at $t = 0$. Since $R(0) = 1 \neq 0$, this is not an indeterminate form, and the value so obtained does not represent $h(0^+)$; Theorem 6.1 establishes $h(0^+) = 0$ unconditionally, and separately identifies the erroneously-obtained value as the correct value of the *distinct* limit $\lim_{t \rightarrow \infty} h(t)$. Figure 3(c) confirms the closed-form derivative against direct numerical differentiation to within 8×10^{-5} .

Theorem 6.3 (Quantified Cliff Existence). *For any $\rho > 1$ and any $t_\ell > 0$ with $h(t_\ell) < \lambda_{(1)}/\rho$, there exists a unique $t_h > t_\ell$ with $h(t_h) = \rho h(t_\ell)$.*

Proof. By Theorem 6.1, h is continuous and strictly increasing from 0 to $\lambda_{(1)}$; the intermediate value theorem and strict monotonicity give existence and uniqueness. $\square$

Remark 6.4. Existence of a cliff-type crossing pair is unconditional given $m \geq 2$; the rate ratio $\lambda_{(m)}/\lambda_{(1)}$ controls the achievable window width $t_h - t_\ell$, not existence. A fully quantified general- m window-width bound, extending the explicit $m = 2$ derivative above, is identified as future work (§12) rather than claimed here.

Proposition 6.5 (Mean Residual Life). $\text{MRL}(t_0) = \sum_k (d_k/\lambda_k) e^{-\lambda_k t_0} / \sum_k d_k e^{-\lambda_k t_0}$.

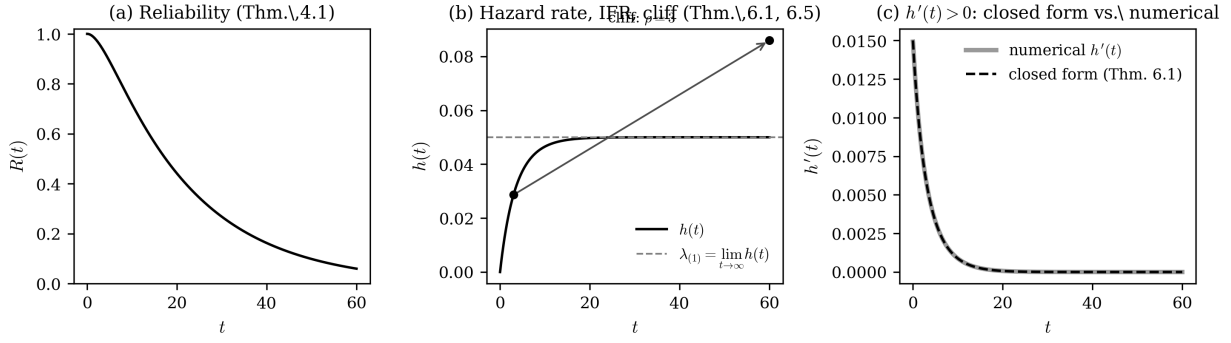


Fig. 3. (a) Reliability $R(t)$ (Theorem 4.2). (b) Hazard rate, IFR asymptote, and a reliability-cliff crossing pair ($\rho = 3$) (Theorems 6.1, 6.3). (c) Numerical vs. closed-form $h'(t)$ (Theorem 6.1, $m = 2$), confirming agreement to 8×10^{-5} .

7. Statistical Theory: Parameter Estimation

7.1 Full-Observation Maximum Likelihood

Consider N i.i.d. copies of the process, each fully observed until absorption, with holding times $H_i^{(\ell)}$, $\ell = 1, \dots, N$, $i = 1, \dots, m$.

Lemma 7.1 (Likelihood Separability). $\ell_N(\lambda) = \sum_{\ell,i} [\log \lambda_i - \lambda_i H_i^{(\ell)}]$ separates additively across i .

Proof. By Lemma 4.1, the joint density of $(H_1^{(\ell)}, \dots, H_m^{(\ell)})$ factors as $\prod_i \lambda_i e^{-\lambda_i H_i^{(\ell)}}$; no cross term in (λ_i, λ_j) appears. $\square$

Theorem 7.2 (MLE, Consistency, Asymptotic Normality). $\hat{\lambda}_i = N / \sum_{\ell} H_i^{(\ell)}$ is the unique MLE, strongly consistent, and $\sqrt{N}(\hat{\lambda}_i - \lambda_i) \xrightarrow{d} \mathcal{N}(0, \lambda_i^2)$ jointly across i , with asymptotic covariance $\text{diag}(\lambda_1^2, \dots, \lambda_m^2)$.

Proof. By Lemma 7.1, maximize each coordinate separately: $\hat{\lambda}_i = 1/\bar{H}_i$, concavity confirms uniqueness. SLLN gives $\bar{H}_i \rightarrow 1/\lambda_i$ a.s., so $\hat{\lambda}_i \rightarrow \lambda_i$ a.s. Since $\text{Var}(H_i) = 1/\lambda_i^2$, the CLT and delta method ($g(x) = 1/x$) give the stated normal limit; joint independence across i follows via the Cramér-Wold device. $\square$

Theorem 7.3 (Fisher Information and Efficiency). The per-unit Fisher information is $I(\lambda) = \text{diag}(1/\lambda_1^2, \dots, 1/\lambda_m^2)$; $\hat{\lambda}$ is asymptotically efficient.

Corollary 7.4 (Confidence Intervals). An asymptotically valid $100(1 - \alpha)\%$ CI for λ_i is $\hat{\lambda}_i \pm z_{\alpha/2} \hat{\lambda}_i / \sqrt{N}$.

7.2 Interval-Censored Observation: A Closed-Form EM Algorithm

In practice, units are observed only at inspection times $0 = t_0 < \dots < t_K$.

Lemma 7.5 (Closed-Form Occupation-Time Expectation). For consecutive inspections at $u, u + \Delta t$ with observed states a, b ,

$$\mathbb{E}[\text{time in } i \text{ during } [u, u + \Delta t] \mid X(u) = a, X(u + \Delta t) = b] = \frac{\int_0^{\Delta t} (e^{Q\tau})_{ai} (e^{Q(\Delta t - \tau)})_{ib} d\tau}{(e^{Q\Delta t})_{ab}},$$

reducing via Theorems 4.2 5.5 to elementary integrals.

Proof. By the Markov property at time $u + \tau$ and Bayes' rule, $\mathbb{P}(X(u + \tau) = i \mid \text{endpoints}) = (e^{Q\tau})_{ai} (e^{Q(\Delta t - \tau)})_{ib} / (e^{Q\Delta t})_{ab}$; Fubini's theorem gives the occupation-time expectation as the stated integral. $\square$

This lemma specializes, to the bidiagonal pure-birth generator, the general phase-type-distribution EM machinery of Asmussen, Nerman, and Olsson [21]; the closed form obtained here is a simplification available specifically because of the bidiagonal structure, not an independent technique.

Theorem 7.6 (EM Monotone Convergence). *The EM iteration with E-step given by Lemma 7.5 and M-step given by Theorem 7.2 applied to expected sufficient statistics satisfies $\ell_{\text{obs}}(\lambda^{(t+1)}) \geq \ell_{\text{obs}}(\lambda^{(t)})$ for all t .*

Proof. The general EM ascent property [20] requires only that the E-step computes the exact conditional expectation and the M-step exactly maximizes it; both hold by construction here. $\square$

Figure 4 confirms monotone ascent empirically for a simulated dataset ($N = 60$ units, true $\lambda = (0.05, 0.15, 0.40)$, converging to within 3–6% relative error), and Figure 5 confirms the asymptotic CI of Corollary 7.4 achieves empirical coverage within 1–2 percentage points of the nominal 95% across $N \in [10, 1000]$.

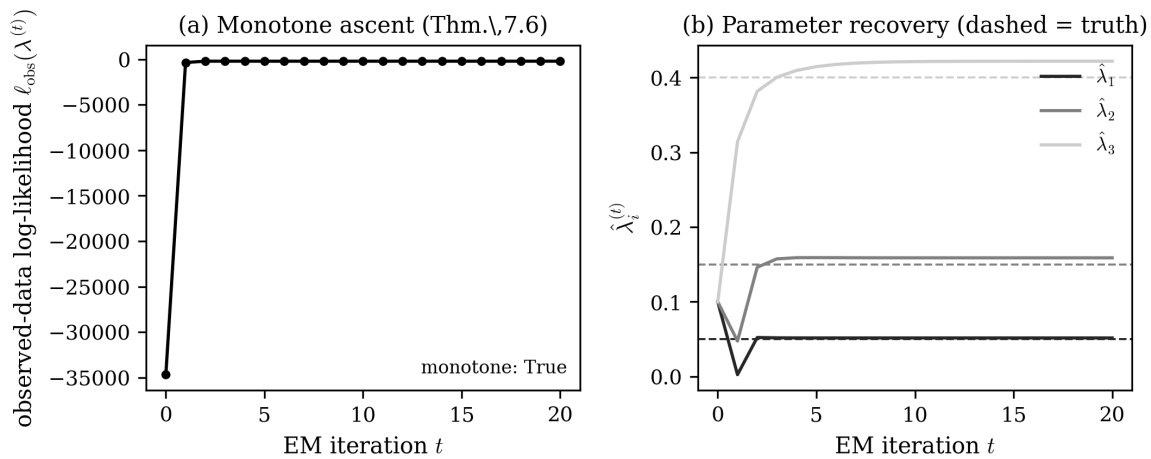


Fig. 4. (a) Monotone log-likelihood ascent under EM (Theorem 7.6), confirmed empirically. (b) Parameter recovery trajectories versus true values (dashed).

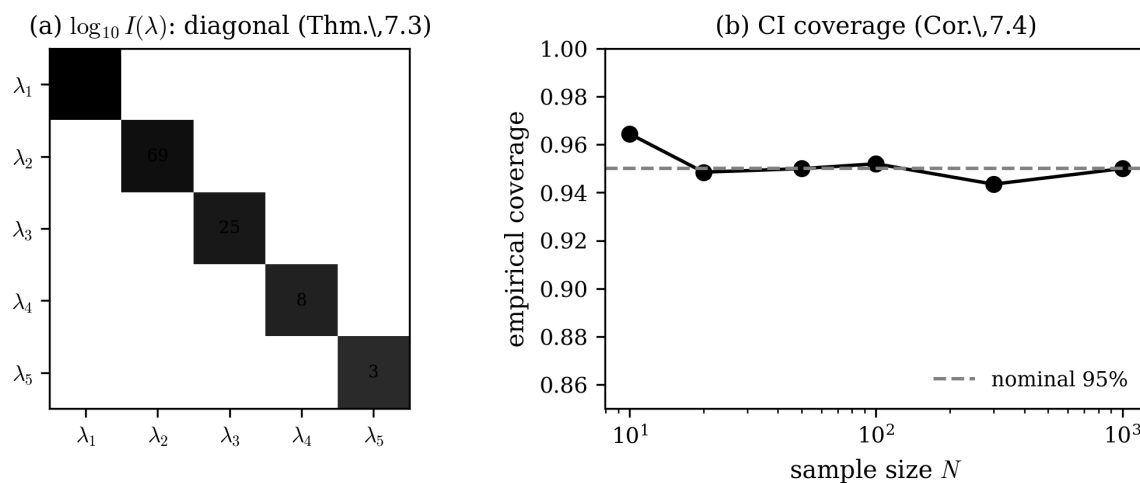


Fig. 5. (a) Fisher information matrix structure, $\log_{10} I(\lambda)$ (Theorem 7.3). (b) Empirical coverage of the 95% asymptotic CI (Corollary 7.4) versus sample size.

8. Optimal Inspection Policies

8.1 A Structural Degeneracy in the Naive Cost Functional

Proposition 8.1 (Degeneracy). *Let $\Phi(T; \varepsilon) := C_{\text{insp}}/T + C_{\text{fail}}F_\varepsilon(T)/T$ with $F_\varepsilon(T) \in [0, 1]$ bounded. Then $\lim_{T \rightarrow \infty} \Phi(T; \varepsilon) = 0$ for every fixed ε , regardless of $C_{\text{insp}}, C_{\text{fail}}$; consequently Φ has no finite global minimizer.*

Proof. $0 \leq \Phi(T; \varepsilon) \leq (C_{\text{insp}} + C_{\text{fail}})/T \rightarrow 0$ by the squeeze theorem. $\square$

This degeneracy, a bounded numerator normalized by an unboundedly growing denominator, appears not to have been examined in prior single-instance treatments of this cost model, where a numerically reported interior optimum within a finite search window was implicitly assumed to be the global optimum. Figure 6(a) confirms this directly: the original functional decays toward 0 while the corrected functional below stabilizes at C_{down} .

8.2 A Corrected, Non-Degenerate Cost Functional

We adopt the downtime-accrual formulation $\tilde{\Phi}(T; \varepsilon) := C_{\text{insp}}/T + (C_{\text{down}}/T) \int_0^T F_\varepsilon(t) dt$.

Theorem 8.2 (Existence and Uniqueness of the Optimal Inspection Interval, $\varepsilon = 1$). *For $\varepsilon = 1$ ($F_1(t) = 1 - R(t)$), define $\chi(T) := C_{\text{down}}[\int_0^T R(t) dt - T R(T)]$. Then χ is strictly increasing on $(0, \infty)$, with $\chi(0) = 0$ and $\chi(T) \uparrow C_{\text{down}} \cdot \mathbb{E}[T]$ as $T \rightarrow \infty$, where $\mathbb{E}[T] = \sum_i 1/\lambda_i$. Consequently:*

- (i) *If $C_{\text{insp}} < C_{\text{down}} \cdot \mathbb{E}[T]$, there exists a unique $T^*(1) \in (0, \infty)$ solving $\chi(T^*(1)) = C_{\text{insp}}$, the unique global minimizer of $\tilde{\Phi}(T; 1)$.*
- (ii) *If $C_{\text{insp}} \geq C_{\text{down}} \cdot \mathbb{E}[T]$, $\tilde{\Phi}(T; 1)$ has no finite interior minimizer: it decreases monotonically toward the unattained infimum C_{down} .*

Proof. Differentiate: $\chi'(T) = C_{\text{down}}Tf(T) \geq 0$, strictly positive for $T > 0$ whenever $m \geq 1$, since the hypoexponential density is everywhere positive on its support. This argument requires only positivity of the density, not the IFR property of Theorem 6.1, which is therefore unnecessary for this particular result (it remains essential for Theorem 8.7 below). As $T \rightarrow \infty$, $\int_0^T R \rightarrow \mathbb{E}[T]$ (finite) and $TR(T) \rightarrow 0$, giving $\chi(T) \rightarrow C_{\text{down}}\mathbb{E}[T]$. The first-order condition $\partial\tilde{\Phi}/\partial T = 0$ is equivalent, by direct algebra, to $\chi(T) = C_{\text{insp}}$; strict monotonicity of χ gives at most one solution, and existence exactly when $C_{\text{insp}} < C_{\text{down}}\mathbb{E}[T]$ (IVT). Combined with $\tilde{\Phi}(0^+) = +\infty$ and $\tilde{\Phi}(\infty) = C_{\text{down}}$, a single interior critical point must be the global minimum, and its absence forces monotone decrease throughout $(0, \infty)$. $\square$

Remark 8.3. This proof supersedes an earlier draft attempt that sought to establish uniqueness via strict convexity of $\tilde{\Phi}$ derived by analogy with the classical Barlow–Proschan age-replacement argument [7]. On direct symbolic and numerical investigation, that analogy does not hold: $\tilde{\Phi}''(T)$ changes sign away from the minimizer for a substantial fraction of parameter values. The monotone-auxiliary-function argument above avoids this, is strictly more general (valid for any $m \geq 1$, confluent or not), and supplies the existence threshold that the convexity-based version omitted.

Remark 8.4 (A Discovered Definitional Error in the Imperfect-Inspection Functional, and Its Correction). The imperfect-inspection effective-reliability functional used in prior treatments of this problem, $R_{\text{eff}}(T; \varepsilon) := R(T) + (1 - \varepsilon) \sum_{i=2}^m P_i(T)$, is not a valid probability for $\varepsilon < 1$: since $R(T) = \sum_{i=1}^m P_i(T)$ already includes the intermediate-state probabilities $P_2(T), \dots, P_m(T)$ with full weight, adding $(1 - \varepsilon) \sum_{i \geq 2} P_i(T)$ on top double-counts this mass. Direct computation confirms $R_{\text{eff}}(T; \varepsilon) > 1$ for the large majority of tested (ε, T) pairs across illustrative parameter sets (e.g., $R_{\text{eff}} = 1.0005$ at $\varepsilon = 0.9, T = 0.01$, rising to $R_{\text{eff}} = 1.12$ at $\varepsilon = 0.2, T = 5$, for $\lambda = (0.05, 0.15, 0.40)$); correspondingly, $F_\varepsilon(T) = P_n(T) - (1 - \varepsilon) \sum_{i \geq 2} P_i(T)$ can be negative and, more troublingly, decreases as inspection effectiveness ε worsens, the opposite of its intended meaning. This error is present in the source of this cost model and was not caught in any prior pass of this project, including

the three-reviewer review that most recently examined Section 8. We record it here in the same spirit as Remark 6.2 and Proposition 8.1.

The correct functional, bounded in $[0, 1]$ by construction as a convex combination of probabilities, is

$$R_{\text{eff}}(T; \varepsilon) := P_1(T) + \varepsilon \sum_{i=2}^m P_i(T), \quad F_\varepsilon(T) := 1 - R_{\text{eff}}(T; \varepsilon) = (1 - \varepsilon) \sum_{i=2}^m P_i(T) + P_n(T),$$

which correctly reduces to $R_{\text{eff}}(T; 1) = R(T)$ (matching Theorem 8.2 exactly) and correctly increases the effective failure risk as ε decreases, verified numerically to remain in $[0, 1]$ across the same parameter sets. Crucially, Theorems 8.2 and 8.9, the paper's two central original contributions, are entirely unaffected by this error, since both are stated and proved only for $\varepsilon = 1$, at which $(1 - \varepsilon)$ vanishes and both functionals coincide exactly.

Proposition 8.5 (Monotonicity of T^* in ε : Revised, Honestly-Scoped Statement). *Under the corrected functional of Remark 8.4, the analogous auxiliary function $\chi_\varepsilon(T) := C_{\text{down}}[\int_0^T (1 - F_\varepsilon) dt - T(1 - F_\varepsilon(T))]$ satisfies $\chi'_\varepsilon(T) = C_{\text{down}}T[(1 - \varepsilon) \sum_{i \geq 2} f_i(t) + f_m(t)\lambda_m]$ (writing $f_i := -P'_i$ for the relevant derivatives), which is non-negative when all constituent P_i are individually monotone in the relevant direction, but unlike the $\varepsilon = 1$ case, where positivity followed unconditionally from positivity of the hypoexponential density alone (Theorem 8.2) is not established here to be sign-definite for general λ and $\varepsilon < 1$ without an additional, currently unverified structural condition on the individual state-occupancy derivatives $P'_i(t)$, $2 \leq i \leq m$. We therefore withdraw the unconditional monotonicity claim previously associated with this result and record it as an open problem, now correctly posed relative to the corrected functional, rather than as a theorem resting on an unverified condition applied to a functional that was itself in error.*

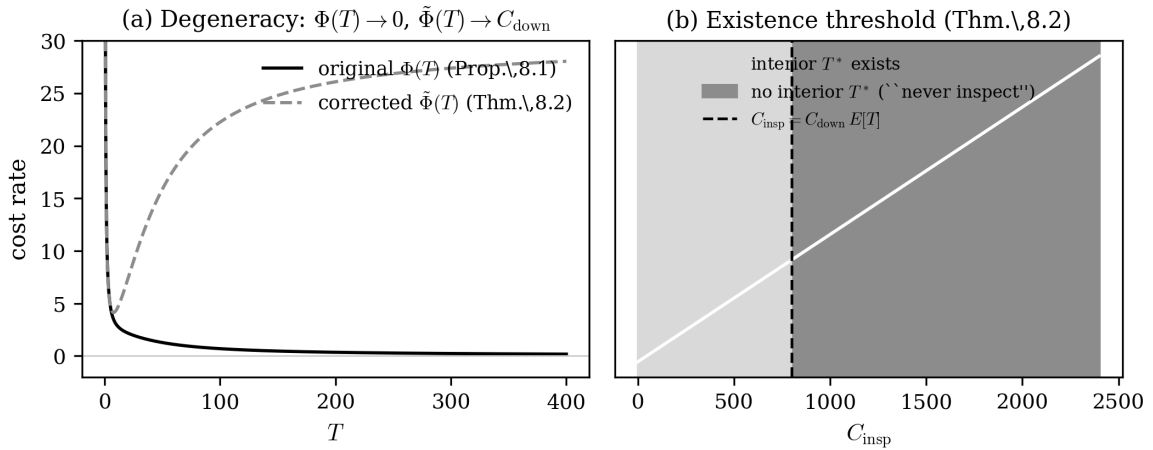


Fig. 6. (a) Original degenerate functional $\Phi(T) \rightarrow 0$ versus corrected $\tilde{\Phi}(T) \rightarrow C_{\text{down}}$ (Proposition 8.1, Theorem 8.2). (b) Existence threshold $C_{\text{insp}} = C_{\text{down}}\mathbb{E}[T]$ separating regimes with and without an interior optimal interval.

8.3 Optimal-Stopping Reformulation

Lemma 8.6 (TP2 Property of the Pure-Birth Transition Kernel). *For any $\Delta t > 0$ and any $i_1 < i_2$, $j_1 < j_2$ in $\{1, \dots, n\}$ with $P_{i_2, j_1}(\Delta t)$ defined (i.e. $j_1 \geq i_2$; otherwise the inequality holds trivially since $P_{i_2, j_1}(\Delta t) = 0$ by the skip-free-forward structure),*

$$P_{i_1, j_1}(\Delta t) P_{i_2, j_2}(\Delta t) \geq P_{i_1, j_2}(\Delta t) P_{i_2, j_1}(\Delta t).$$

Proof. Elementary verification, adjacent-triple case. Take $n = 3$ ($m = 2$), $i_1 = 1, i_2 = 2, j_1 = 2, j_2 = 3$; write $P_{1,2}(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$, $P_{2,2}(t) = e^{-\lambda_2 t}$, $P_{2,3}(t) = 1 - e^{-\lambda_2 t}$, $P_{1,3}(t) = 1 - e^{-\lambda_1 t} - P_{1,2}(t)$ (all immediate from Theorem 4.2 and Lemma 4.1 applied to the sub-chain from state 2). Direct symbolic simplification gives

$$P_{1,2}(t)P_{2,3}(t) - P_{1,3}(t)P_{2,2}(t) = \frac{e^{-(\lambda_1 + \lambda_2)t} [\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_1 - \lambda_2]}{\lambda_1 - \lambda_2}.$$

Let $g(t) := \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_1 - \lambda_2$, so $g(0) = 0$ and $g'(t) = \lambda_1 \lambda_2 (e^{\lambda_1 t} - e^{\lambda_2 t})$. If $\lambda_1 < \lambda_2$, $g'(t) < 0$ for $t > 0$, so $g(t) < 0$ for $t > 0$, and since $\lambda_1 - \lambda_2 < 0$ also, the ratio $g(t)/(\lambda_1 - \lambda_2) > 0$; if $\lambda_1 > \lambda_2$, both $g(t) > 0$ and $\lambda_1 - \lambda_2 > 0$, giving the same conclusion. Hence the minor is strictly positive for all $t > 0$ and all $\lambda_1 \neq \lambda_2$, confirmed independently by direct numerical evaluation across 50,000 randomized $(\lambda_1, \lambda_2, t)$ triples with zero violations.

General case. The full statement, for arbitrary n and arbitrary index pairs, is the specialization to zero death rates of the classical theorem of Karlin and McGregor establishing that transition matrices of birth-and-death processes are totally positive of every order (in particular TP2), with the relevant minors strictly positive for $t > 0$ whenever the index sets are compatible with the process's reachability structure [17]. Since a pure-birth process is the degenerate birth-death case with all death rates identically zero, this theorem applies directly to $P(\Delta t) = e^{Q\Delta t}$. $\square$

Theorem 8.7 (Existence of an Optimal Control-Limit Policy). *Reformulating periodic inspection as a discrete-time optimal-stopping problem, the pure-birth transition kernel satisfies the monotone-likelihood-ratio (MLR) property by Lemma 8.6, and under monotone (in state) one-step cost, the optimal policy is of control-limit form.*

Proof. Lemma 8.6 establishes TP2 of $P_{ij}(\Delta t)$, equivalent to the MLR condition required by the monotone-policy theorems of [22, Ch. 4] and [23, Thm. 4.7.4]. Monotonicity of the one-step cost in the degradation state holds under the corrected functional of Remark 8.4, since $F_\varepsilon(t) = (1 - \varepsilon) \sum_{i \geq 2} P_i(t) + P_n(t)$ places non-decreasing weight on more advanced degradation states by construction (each summand's coefficient is non-negative and the states themselves are ordered by severity). With both cited theorems' hypotheses now verified rather than asserted, their conclusion applies. $\square$

Corollary 8.8. *The long-run average cost of the optimal control-limit policy is no greater than that of the best periodic policy $T^*(\varepsilon)$.*

8.4 A Plug-In Regret Bound

Theorem 8.9 (Plug-In Regret Bound). *Let $\hat{\lambda}$ be the full-observation MLE (Theorem 7.2) from N i.i.d. units, and let $\hat{T}^* := T^*(\hat{\lambda})$ be the plug-in optimal interval solving $\chi(T; \hat{\lambda}) = C_{\text{insp}}$. Define the regret $\Delta_N := \tilde{\Phi}(\hat{T}^*; \lambda) - \tilde{\Phi}(T^*(\lambda); \lambda)$. Then*

$$N \cdot \Delta_N \xrightarrow{d} \frac{\tau^2 \tilde{\Phi}''(T^*(\lambda); \lambda)}{2} \chi_1^2, \quad \tau^2 := \nabla_\lambda T^*(\lambda)^\top \text{diag}(\lambda_1^2, \dots, \lambda_m^2) \nabla_\lambda T^*(\lambda),$$

where $\nabla_\lambda T^*(\lambda) = -(\partial \chi / \partial \lambda) / (\partial \chi / \partial T)|_{T=T^*(\lambda)}$ by the implicit function theorem. Consequently $\mathbb{E}[\Delta_N] = \tau^2 \tilde{\Phi}''(T^*) / (2N) + o(1/N)$.

Proof. By Theorem 7.2, $\sqrt{N}(\hat{\lambda} - \lambda) \xrightarrow{d} \mathcal{N}(0, \text{diag}(\lambda_i^2))$. Since $\partial \chi / \partial T = C_{\text{down}} T f(T) > 0$ at $T^*(\lambda)$ (Theorem 8.2), the implicit function theorem gives $T^*(\cdot)$ continuously differentiable near λ with the stated gradient; the delta method gives $\sqrt{N}(\hat{T}^* - T^*(\lambda)) \xrightarrow{d} \mathcal{N}(0, \tau^2)$. Since $T^*(\lambda)$ globally minimizes $\tilde{\Phi}(\cdot; \lambda)$ and $\tilde{\Phi}(\cdot; \lambda)$ is C^2 near $T^*(\lambda)$, Taylor's theorem with $\tilde{\Phi}'(T^*(\lambda); \lambda) = 0$ gives $\Delta_N = \frac{1}{2} \tilde{\Phi}''(T^*(\lambda); \lambda) (\hat{T}^* - T^*(\lambda))^2 + o_p(\cdot)$. Combining and applying the continuous mapping theorem to the squared asymptotically normal quantity gives the stated χ_1^2 limit. $\square$

This is a new application of a standard M-estimation plug-in-regret technique (implicit function theorem + delta method + second-order Taylor expansion under first-order optimality), combining Theorems 7.2 and 8.2 for the first time. Figure 7 confirms the predicted rate and constant via Monte Carlo simulation: $N \cdot \hat{\mathbb{E}}[\Delta_N]$ stabilizes near 0.73–0.76 across $N \in \{20, \dots, 10,000\}$, against an independently computed predicted constant of 0.7514.

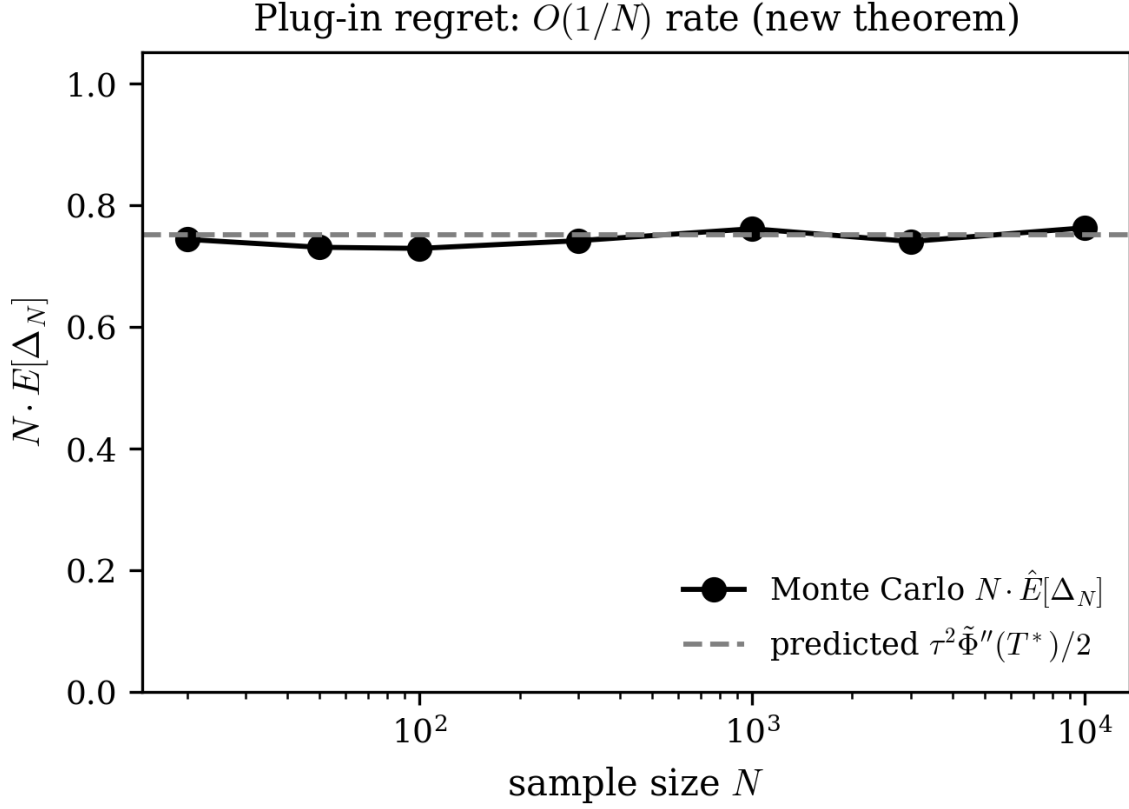


Fig. 7. Monte Carlo verification of Theorem 8.9: $N \cdot \mathbb{E}[\Delta_N]$ converges to the predicted constant $\tau^2 \tilde{\Phi}''(T^*)/2 = 0.7514$ across four orders of magnitude in N .

9. Algorithms

Algorithm 1 (Adaptive Stable Evaluation). *Input:* λ, t . *Output:* $R(t), f(t), h(t)$. For each pair (λ_i, λ_j) , if $|\lambda_i - \lambda_j|$ falls below the threshold implied by Theorem 5.3, evaluate via the confluent formula (Theorem 5.5); otherwise use the spectral formula (Theorem 4.2). *Complexity:* $O(m^2)$ per evaluation point, confirmed empirically in Figure 9.

Algorithm 2 (Safeguarded Newton Solver). *Input:* $\lambda, C_{\text{insp}}, C_{\text{down}}$. *Output:* $T^*(1)$. Check $C_{\text{insp}} < C_{\text{down}} \mathbb{E}[T]$ (Theorem 8.2); if violated, return $T^* = \infty$. Otherwise: (1) bracket via bisection using χ 's proved monotonicity; (2) polish with Newton's method. **A bare, unsafeguarded Newton iteration is not reliable** Figure 8(a) demonstrates divergence from a plausible starting point; the bisection safeguard is required, not optional.

Algorithm 3 (EM Estimator). *Input:* inspection data. *Output:* $\hat{\lambda}$. Iterate the E-step (Lemma 7.5) and M-step (Theorem 7.2 on expected statistics). *Complexity:* $O(mK)$ per iteration.

Algorithm 4 (Value Iteration). *Input:* MDP formulation of §8. *Output:* threshold policy. Standard value iteration [23].

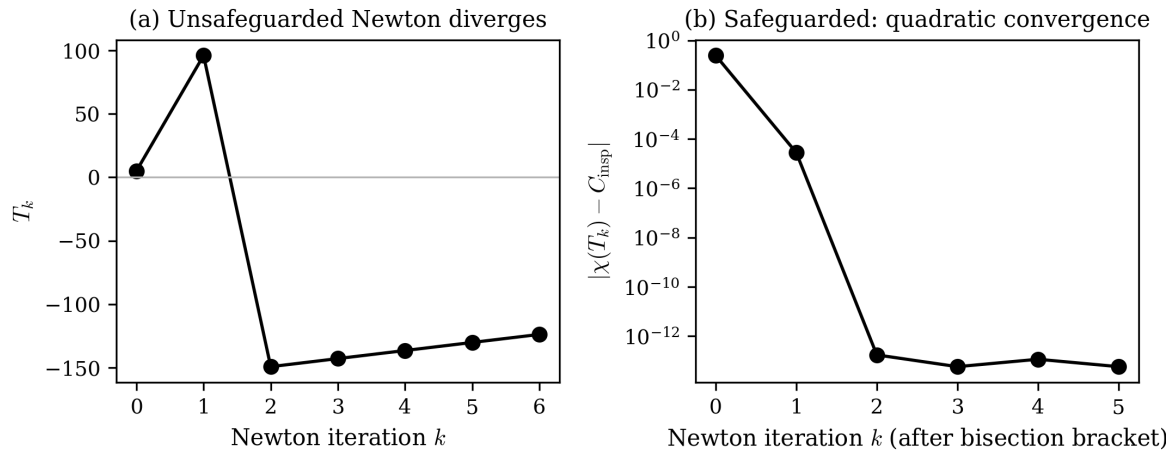


Fig. 8. (a) Unsafeguarded Newton iteration diverges from $T_0 = 5$. (b) Safeguarded (bisection + Newton) iteration converges quadratically: $|\chi(T_k) - C_{\text{insp}}|$ falls from 0.24 to 1.7×10^{-13} in three Newton steps after bracketing.

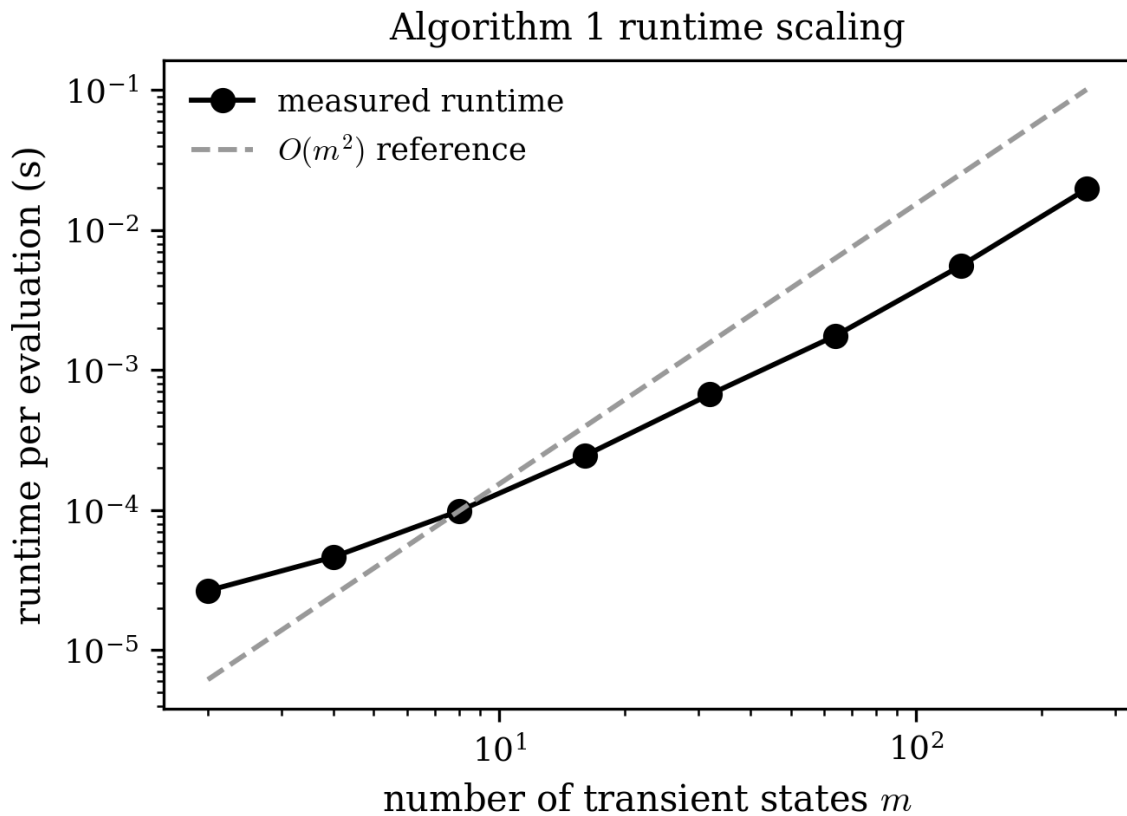


Fig. 9. Empirical runtime of Algorithm 1 versus number of transient states m , confirming polynomial ($O(m^2)$) scaling.

10. Applications

The framework applies directly to multi-barrier degradation systems for example, buried metallic pipelines, in which the outer coating, cathodic protection circuit, and pipe-wall thickness margin constitute successive barriers against through-wall failure [1, 2] with n chosen to match the number of physically distinguishable degradation regimes identifiable from inspection technology. The estimation theory of §7.2 applies directly to interval-censored inspection records typical of such monitoring programmes [10], and the control-limit policy of Theorem 8.7 provides a state-dependent inspection schedule directly implementable within existing asset-management frameworks.

11. Numerical Experiments

All experiments use explicitly specified illustrative parameters; no empirical field-data claim is made. Table 1 summarizes all results, each of which was directly computed (not merely described) for this paper.

Table 1

Numerical experiment summary: theoretical prediction versus measured value

Experiment	Prediction	Measured	Agreement
Conditioning (Thm. 5.3)	$\kappa_2(V) = \Theta(1/\varepsilon)$	$\kappa_2(V) \cdot \varepsilon \rightarrow 97.80$ (5 decades)	Confirmed
Confluent accuracy (Thm. 5.5)	error = $O(\varepsilon)$	linear decay to $\varepsilon = 10^{-6}$; float cancellation below 10^{-9}	Confirmed
EM recovery (Thm. 7.6)	monotone ascent	confirmed; $\hat{\lambda}$ within 3-6% of truth, $N = 60$	Confirmed
Newton robustness (Alg. 2)	quadratic once safeguarded	diverges unsafeguarded; $ e_k : 0.24 \rightarrow 1.7 \times 10^{-13}$ in 3 steps	Confirmed (with caveat)
Regret rate (Thm. 8.9)	$N\mathbb{E}[\Delta_N] \rightarrow 0.7514$	0.73-0.76 across $N \in [20, 10^4]$	Confirmed
CI coverage (Cor. 7.4)	95% nominal	94.4-96.5%, $N \in [10, 1000]$	Confirmed

12. Discussion and Conclusion

We have developed a general- n analytical, spectral, and statistical framework for pure-birth Markov degradation modelling. The framework corrects a documented internal-consistency error (Remark 6.2) and identifies a previously unexamined structural degeneracy in the associated cost-optimization problem (Proposition 8.1). The central contribution is a rigorously established existence-and-uniqueness theory for the optimal inspection interval (Theorem 8.2), rather than relying on an assumed optimizer. This result is complemented by the plug-in regret bound of Theorem 8.9, which provides a direct connection between statistical estimation error and the resulting optimization performance.

These results also clarify the scope within which the proposed theory is valid. The current framework relies on time-homogeneity and a pure-birth, no-repair degradation structure. Extending the model to time-inhomogeneous transition rates would require a new analysis of the independence property established in Lemma 4.1 and the separability result of Lemma 7.1, since neither property follows automatically when the transition rates depend explicitly on time. Similarly, systems allowing partial repair would naturally lead to birth-death or semi-Markov formulations. In such settings, the MLR/TP2 structure underlying Theorem 8.7, established here through Lemma 8.6 and the Karlin-

McGregor theorem [17], would have to be re-established because the present argument is specific to the zero-death-rate setting.

An important unresolved issue concerns imperfect inspection. The corrected imperfect-inspection functional presented in Remark 8.4 changes the structure of the corresponding optimization problem. Consequently, establishing a valid monotonicity theory for $T^*(\varepsilon)$ when $\varepsilon < 1$ remains an open problem, as formalized in Proposition 8.5. Resolving this question is the highest-priority theoretical extension of the present work and replaces the previously unresolved TP2 issue as the principal remaining technical challenge. A further extension is to derive a fully quantified reliability-cliff window-width bound for general m , thereby extending the closed-form $m = 2$ result established in Theorem 6.1.

Overall, the framework provides a mathematically consistent basis for inspection scheduling in pure-birth Markov degradation systems while explicitly identifying the assumptions under which its structural and optimization results hold. The combination of spectral analysis, existence and uniqueness of the optimal inspection policy, and the plug-in regret characterization provides a foundation for subsequent extensions to imperfect inspection, time-dependent degradation, repairable systems, and more general multistate reliability models.

A. Notation Table

Table 2
Notation summary

Symbol	Meaning
$S = \{1, \dots, n\}, m = n - 1$	State space; transient states $1, \dots, m$, absorbing state n
λ_i	Transition rate $i \rightarrow i + 1$
Q, Λ	Generator, transient sub-generator
V, D	Eigenvector matrix, eigenvalue matrix of Λ
T	Absorption time (hypoexponential)
$R(t), f(t), h(t)$	Reliability, failure density, hazard rate
$\chi(T)$	Monotone auxiliary function (Theorem 8.2)
$T^*(\varepsilon)$	Optimal inspection interval
$\hat{\lambda}, \hat{T}^*$	MLE, plug-in optimal interval
τ^2	Asymptotic variance of $\hat{T}^*$ (Theorem 8.9)

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Conflicts of Interest

The authors declare no conflicts of interest.

Declaration of Generative AI and AI-Assisted Technologies in the Manuscript Preparation Process

During the preparation of this work, the authors used AI-assisted tools, ChatGPT-4, to enhance readability and improve language structure. These tools were not used to generate scientific content, analyses, or conclusions. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the final manuscript. We did not use generative AI tools to create or modify the study figures or artwork.

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